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Multigrid algorithms in divergence free space for Stokes problem

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In this paper, We present a new prolongation operator for the divergence-free part of a P2-P1 discretization of the two dimensional Stokes problem. Also, the convergence of multigrid algorithm would be proved for this discretization. Moreover, it is shown that this prolongation operator will preserve the discrete divergence-free property.
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1 Introduction

Multigrid algorithms have been used extensively as tools for obtaining approximations to the solutions of partial differential equations. In conjunction, there has been intensive research into the theoretical understanding these methods ([1], [4], [7]). Multigrid methods for finite elements is extensively applicable in the all of sciences such as electricity, fluid mechanics and so on (see [9] for instance). We consider the two dimensional Stokes problem in a simply connected bounded polygonal domain $\Omega \subset R^2$ with homogeneous Dirichlet boundary condition:

$$\begin{array}{rcll} -\Delta \mathbf{u} + \nabla p &=& \mathbf{f} & in \quad \Omega, \\ \nabla . \mathbf{u} &=& 0 & in \quad \Omega, \\ \mathbf{u} &=& \mathbf{0} & on \quad \partial\Omega. \end{array} \qquad (1)$$

The equation $\nabla . \mathbf{u} = 0$ in (1) expresses the incompressibility condition. The variational form of (1) is to find $\mathbf{u} = (u_1, u_2)^T \in V = (H_0^1(\Omega))^2$ and $p \in M = L_2(\Omega)$ such that

$$\begin{array}{rcll} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &=& (\mathbf{f},\mathbf{v})_0 & \forall \mathbf{v} \in V, \\ b(\mathbf{u},q) &=& 0 & \forall q \in M, \end{array} \qquad (2)$$

in which $(.,.)_0$ denotes an inner product in the $L_2(\Omega)$ space. Also, the bilinear forms $a(.,.)$ on $V \times V$ and $b(.,.)$ on $V \times M$ are given respectively by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla \mathbf{u} . \nabla \mathbf{v} d\mathbf{x},$$

$$b(\mathbf{u},p) = -\int_\Omega p \nabla . \mathbf{u} d\mathbf{x}.$$

Theoretically, we can use the second equation in (2) (i.e. incompressibility condition) to define the divergence free subspace

$Z = \{\mathbf{v} \in V : b(\mathbf{v},q) = 0, \quad \forall q \in M\} \subset V.$

Then, the system (2) splits to a Z-elliptic problem:

Find $\mathbf{u} \in Z$ such that

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_0 \quad \forall \mathbf{v} \in Z \qquad (3)$$

and a subsequent problem for the pressure $p \in M$:

$$b(\mathbf{v},p) = (\mathbf{f},\mathbf{v})_0 - a(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in V. \qquad (4)$$

Then, for solving (2), at first with eliminating p, the system (3) is solved to determine the velocity $\mathbf{u}$ and then with solving (4), the pressure p is obtained.

For discrete mixed formulation of (2), let $V_h \subset V$ and $M_h \subset M$ and consider the variational problem to find $\mathbf{u_h} \in V_h$ and $p_h \in M_h$ such that

$$\begin{array}{c} a(\mathbf{u_h},\mathbf{v_h}) + b(\mathbf{v_h},p_h) = (\mathbf{f},\mathbf{v_h})_0, \\ \forall \mathbf{v_h} \in V_h, \\ b(\mathbf{u_h},q_h) = 0, \forall q_h \in M_h. \end{array} \qquad (5)$$

Similarly, define

$$Z_h = \{\mathbf{v_h} \in V_h : b(\mathbf{v_h}, q_h) = 0, \quad q_h \in M_h\} \subset V_h,$$

where it is called the subspace of discretely divergence free velocities in V_h. Note that often, Z_h is not a subspace of Z (see [6]). Then (5) is equivalent to the following: Find $\mathbf{u_h} \in Z_h$ such that

$$a(\mathbf{u_h}, \mathbf{v_h}) = (\mathbf{f}, \mathbf{v_h})_0 \quad \forall \mathbf{v_h} \in Z_h, \quad (6)$$

and then determining $p_h \in M_h$ such that
for each $\mathbf{v_h} \in V_h$

$$b(\mathbf{v_h}, p_h) = (\mathbf{f}, \mathbf{v_h})_0 - a(\mathbf{u_h}, \mathbf{v_h}). \quad (7)$$

The following Inf-Sup Condition will guarantee the uniqueness of the solution of (5).

Lemma 1 *In order for (7) to have a unique solution, it is necessary and sufficient that*

$$0 < \inf_{q_h \in M_h} \sup_{\boldsymbol{v_h} \in V_h} \frac{|b(\boldsymbol{v_h}, q_h)|}{\| \boldsymbol{v_h} \|_V \| q_h \|_M}.$$

Proof: See ([14]).

In Section 2, we will describe the basis functons for Taylor- Hood triangular elements that have been originally introduced by Hecht. In Section 3, we will introduce a new prolongation operator and then the symmetric multigrid algorithm is expressed. Moreover, the convergence of multigrid algorithm would be proved. In Section 4, some numeical results are given and finally in Section 5, a summary of this paper and some suggestions for future research are given.

2. Triangular Taylor-Hood Elements

We consider the P2-P1 Taylor-Hood element
for the discretization of (1) on a triangulation of a simply connected polygonal domain $\Omega \subset R^2$ as our choices for V and M. There is a local basis of Z that this local basis of discretely divergence free (ddf) space has been originally introduced by Hecht in his paper. These basis functions are the following (for more details see [13]):

1) The vertex basis functions $\mathbf{v_{p,1}}$ and $\mathbf{v_{p,2}}$ that are (1,0) and (0,1) on vertex p respectively and zero on the other nodal points(vertices and edge midpoints). Figure (1) shows the nodal points and the supports of these vertex basis functions.

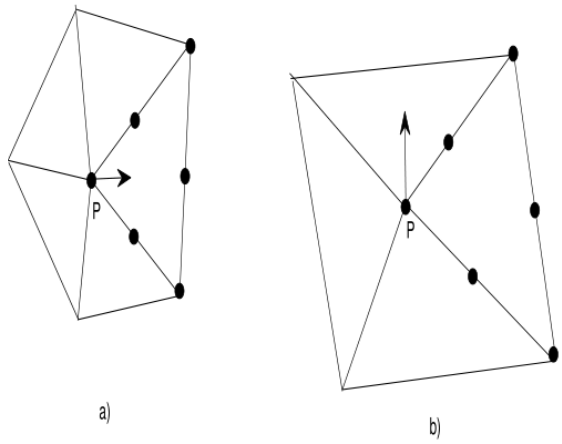

Figure 1: Vertex basis functions $\mathbf{z_{p_1}} = \mathbf{v_{p_1}}$ and $\mathbf{z_{p_2}} = \mathbf{v_{p_2}}$.

2) Let us consider the following vectors:

$$\mathbf{v_{e,t}}(Q) = \begin{cases} C_e\mathbf{e} & \text{if } \mathrm{Q} = \mathrm{M_e}, \\ \mathbf{0} & \text{if } \mathrm{Q} \neq \mathrm{M_e}, \end{cases}.$$

and

$$\mathbf{v_{e,d}}(Q) = \begin{cases} C_e\mathbf{d} & \text{if } \mathrm{Q} = \mathrm{M_e}, \\ \mathbf{0} & \text{if } \mathrm{Q} \neq \mathrm{M_e}, \end{cases}.$$

where M_e is the midpoint of interior edge e, the vectors $\mathbf{e}$ and $\mathbf{d}$ are two tangential and diagonal edge vectors of the quadrilateral given by the two triangles attached to edge e (see Figure (2))and C_e is a normalization constant.

Now, for any two (interior) neighbor triangles (a triangle is called interior if all its edges are interior, otherwise it is a boundary triangle), a basis function in ddf space is the summation of the vectors $\mathbf{v_{e,t}}$ and $\mathbf{v_{e,d}}$. In fact, the summation of input and output fluxes should be equal. Figures 3(a) to 3(d) illustrate these basis functions. In Figures 3(a) and 3(b), a local ddf basis function is $\mathbf{z_e} = \mathbf{v_{e_1,t}} + \mathbf{v_{e_2,t}} + \mathbf{v_{e_3,d}}$ and also in Figures 3(c) and 3(d), a local ddf basis function is $\mathbf{z_e} = \mathbf{v_{e_1,t}} +$

$\mathbf{v}_{\mathbf{e}_2,\mathbf{t}} + \mathbf{v}_{\mathbf{e}_3,\mathbf{t}} + \mathbf{v}_{\mathbf{e}_4,\mathbf{d}}$.

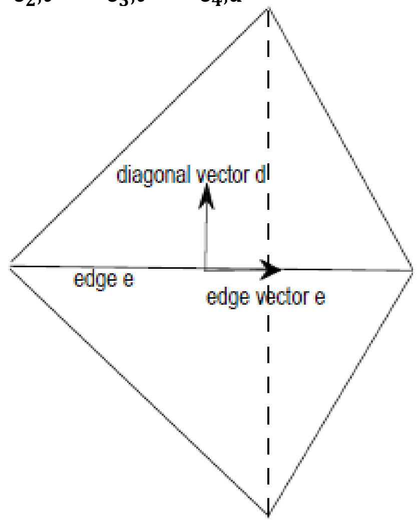

Figure 2: Tangential and diagonal edge vectors.

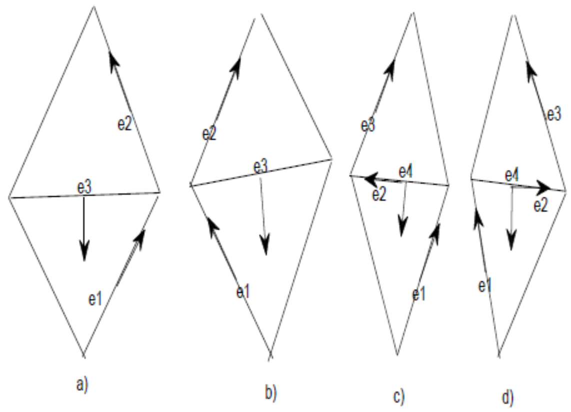

Figure 3: Local basis functions $\mathbf{z}_{\mathbf{e}}$.

3) In each interior triangle, the summation of the vector functions $\mathbf{v}_{\mathbf{e},\mathbf{t}}$ considering their directions contain a ddf triangular basis function. In Figure (4), a local triangular basis function is $\mathbf{z}_{\Delta} = \mathbf{v}_{\mathbf{e}_1,\mathbf{t}} + \mathbf{v}_{\mathbf{e}_2,\mathbf{t}} + \mathbf{v}_{\mathbf{e}_3,\mathbf{t}}$.

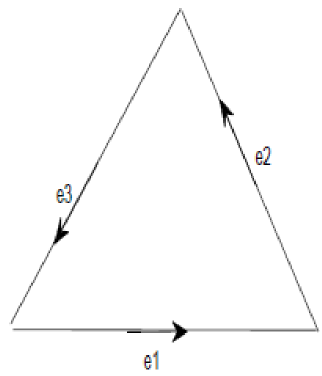

Figure 4: Local triangular basis function $\mathbf{z}_{\Delta}$.

Now, we are ready to state the following theorem:

Theorem 2 *The space $\mathcal{Z}$ of ddf velocities for the triangular P2-P1 Taylor-Hood element with respect to a triangulation on a simply connected polygonal domain Ω possesses a basis*

$$\mathcal{Z} = \mathcal{Z}_p \cup \mathcal{Z}_e \cup \mathcal{Z}_\Delta$$

of locally supported functions where $\mathcal{Z}_p$ consists of all vertex basis functions $Z_{p,k}$, $\mathcal{Z}_e$ of a selection of diagonal edge functions $\mathbf{z}_{e,\mathbf{d}}$ as specified above and $\mathcal{Z}_\Delta$ of all triangle functions $\mathbf{z}_{\Delta}$ associated with interior Δ.

Proof: See [13].

3 Multigrid Methods on Triangular Taylor-Hood Elements

Solving the Stokes problem on triangular Taylor- Hood elements has been verified by some researchers (see e.g. [6], [12]). Furthermore, in [13] the Stokes problem has been solved by the same basis functions given in Section 2. In [13], Oswald has proposed a specific preconditioning method for solving the Stokes problem. However, as it has been mentioned in [13], this preconditioning method is not suitable for large scale. In this section, by introducing a new prolongation operator, we want to solve the Stokes problem by multigrid algorithms on triangular Taylor- Hood elements. This method also works for large scales. Moreover, it is shown that under some conditions, the multigrid algorithm converges. Up to now, a lot of prolongation operators have been constructed for conforming and nonconforming finite element discretizations (see e.g. [8], [12], [11] for some intergrid transfer operators). The construction of a prolongation operator for Whitney elements on simplicial meshes has been given in [3]. Also, in [10] a prolongation and restriction operator has been given for mixed finite element discretizations of the generalized Stokes problem using the Scott- Vogelius element. In [15], new second-order prolongation and restriction formulas have been given which preserve the divergence and, in some cases, the curl of a discretized vector field. Furthermore, a divergence free prolongation operator in [5] has been applied to estimate the magnetic field in the refined cells for astrophysical MHD. Now, we are ready to introduce our method for solving the Stokes problem on triangular Taylor-Hood elements. The basis functions are the same functions given in Section 2. We note that the ddf space in the level j (i.e. $Z_j = \{\mathbf{v} \in V_j : b(\mathbf{v}, q) = 0, \quad \forall q \in M_j\} \subset V_j$) is not the subspace of ddf space in the level $j+1$ (i.e. Z_{j+1}). Then the usual prolongation operators would not be satisfied in this method. To develop a multigrid algorithm for the discretization problem (6), we need to assume a structure to our family of partitions. For $0 < h < 1$, let Γ_h be a triangulation of Ω into triangles of size h. Now, let h_1 and $\Gamma_{h_1} = \Gamma_1$ be given. For each integer $1 < j \leq J$, let $h_j = 2^{1-j} h_1$ and $\Gamma_{h_j} = \Gamma_j$ be constructed by connecting the midpoints of the edges of the triangle in Γ_{j-1} and let $\Gamma_h = \Gamma_J$ be the finest grid. In this and the following section, we replace subscript h_j simply by subscript j.

3.1 Intergrid Transfer Operators

For construction of prolongation operator, we define the coarse to fine intergrid transfer operator $I_j : Z_{j-1} \to Z_j$ for $j = 2,3,\cdots,J$ by $I_j = I_{j_1} O I_{j_2}$ where $I_{j_1} : V_j \to Z_j$ and $I_{j_2} : Z_{j-1} \to V_j$ are defined as follows. To define $I_{j_2} : Z_{j-1} \to V_j$, we first recall that if P is at most a quadratic polynomial on a triangle T in Γ_{j-1} such that $P(p_i) = f_i$ for $i = 1,2,3$ and $P(M_i) = m_i$ where p_i and M_i for $i = 1,2,3$ are the vertices and midpoints of T respectively, then

$$P(Q_1) = \frac{m_1 + m_2}{2} + \frac{m_3}{3} - \frac{f_1 + f_2}{8} \qquad (8)$$

and similarly holds for Q_2 and Q_3 (see Figure 5.a).

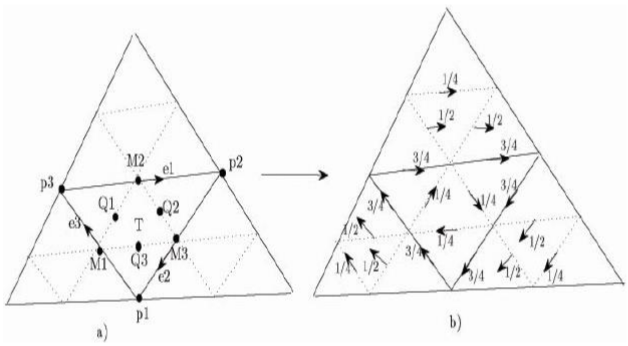

Figure 5: a) Local triangle basis function $\mathbf{z}_\Delta$ in coarse grid. b) Effect of $\mathbf{z}_\Delta$ on fine grid.

Now, let us to consider a local triangular basis function $\mathbf{z}_\Delta = \mathbf{v}_{\mathbf{e_1},\mathbf{t}} + \mathbf{v}_{\mathbf{e_2},\mathbf{t}} + \mathbf{v}_{\mathbf{e_3},\mathbf{t}}$ (see Figure 5.a). Let Q be a midpoint of an edge of an arbitrary triangle Δ in Γ_j; then we can define $I_{j_2}\mathbf{z}_\Delta$ by (8),

$$(I_{j_2}\mathbf{z}_\Delta)(Q) = \begin{cases} \mathbf{0} & \text{if } \mathrm{Q} \in \partial\Omega, \\ \frac{3}{4}\mathbf{e_i} & \text{if } \mathrm{Q} \in \Gamma_{\mathrm{j-1}} \text{ and Q on } \mathrm{e_i}, \\ \frac{\mathbf{e_1} + \mathbf{e_3}}{2} + \frac{\mathbf{e_2}}{4}, & \text{if } \quad \mathrm{Q} = \mathrm{Q_1} (\text{ Figure 5}), \end{cases}$$

where $\mathbf{e_i}$ is the tangential vector on the edge e_i.

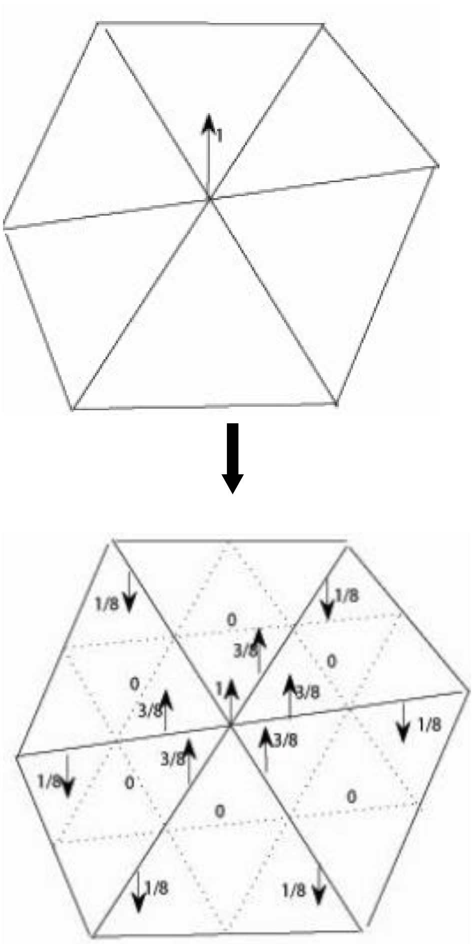

Figure 6: Mapping the $I_{j_2}\mathbf{V_{p,2}}$.

Similar relations hold when $Q = Q_2$ or $Q = Q_3$. In addition, as Figure (5) shows, $(I_{j_2}\mathbf{z_\Delta})(Q) = 1/2\mathbf{e_i}$ and $(I_{j_2}\mathbf{z_\Delta})(Q) = 1/4\mathbf{e_i}$ so that Q is within the adjacent triangle of $T \in \Gamma_j$ and on one side of the edge e_i. Also, if Q is a midpoint of an edge e_i of T in Γ_{j-1}, then $(I_{j_2}\mathbf{z_\Delta})(Q) = \mathbf{e_i}$. Furthermore, let $\mathbf{v_{p,1}}$ and $\mathbf{v_{p,2}}$ be the vertex basis functions corresponding the vertex p of a triangle T in Γ_{j-1}. If Q is a vertex of a triangle in Γ_j, then we can simply define

$$(I_{j_2}\mathbf{v_{p,i}})(Q) = \begin{cases} \mathbf{p_i}, & \text{if } \mathrm{Q = p}, \\ \mathbf{0}, & \text{if } \mathrm{Q \neq p}, \end{cases}$$

for $i = 1,2$ which $\mathbf{p_i}$ is the unit vector with 1 in ith component. Similarly, by formula (8) the mapping $I_{j_2}: Z_{j-1} \to V_j$ is defined for other local basis functions (see Figure 6 for instance).

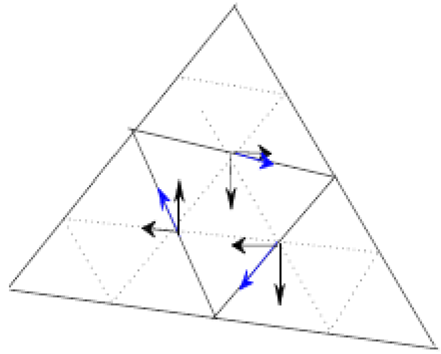

Figure 7. Effect of local triangle basis function in level $j-1$ on the vertex basis functions in the level j.

To define the mapping $I_{j_1}: V_j \to Z_j$, we consider three cases:

Case 1: we note that the vertex basis functions $\mathbf{v_{p,1}}$ and $\mathbf{v_{p,2}}$ belong to the ddf space. Then we can simply define $I_{j_1}\mathbf{p_i} = \mathbf{p_i}$ for $i = 1,2$. **Case 2:** The mapping I_{j_1} projects each vector $\mathbf{e} = (e_1, e_2)^T$ on the middle point of an edge in Γ_{j-1} to horizontal and vertical components (Figure 7), i.e.

$$I_{j_1}\mathbf{e} = \begin{cases} e_1\mathbf{p_1}, \\ e_2\mathbf{p_2}. \end{cases}$$

Case 3: To make the mapping I_{j_1} on the midpoint of an edge of Δ in Γ_j like Q, we have to note that several local basis functions usually affect on Q. For example, as Figure 8 shows, three local basis functions $\mathbf{z_{e,d_1}}, \mathbf{z_{e,d_2}}, \mathbf{z_{e,d_3}}$ and one local triangle basis function $\mathbf{z_\Delta}$ affect on Q. Then, in order to obtain the value of $\frac{3}{4}\mathbf{e1}$ on Q (see Figure 5.b), we can form the following equation:

$$\mathbf{z_{e,d_1}} + \mathbf{z_{e,d_2}} + \mathbf{z_{e,d_3}} + \mathbf{z_\Delta} = \frac{3}{4}\mathbf{e_1}.$$

In this manner, for each vector function in V_j, similar equations can be easily formed on other midpoints of an edge in the fine grid Γ_j. Therefore, a system of Ax=b is formed and thus the coefficients of ddf basis functions in the fine grid are determined.

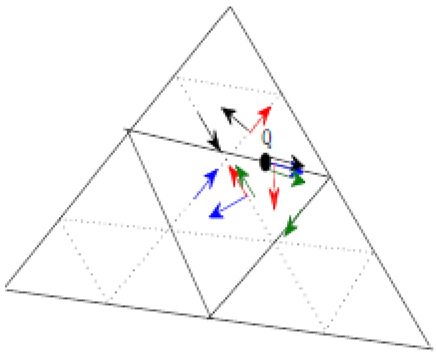

Figure 8. Three edge and one triangular local basis functions on the point Q.

We observe that this prolongation operator will preserve the discrete divergence free property from level $j-1$ to j. The restriction operator $R_j: Z_j \to Z_{j-1}$ can be defined by $R_j = C(I_j)^T$ where C is a constant.

3.2 Multigrid Methods

In this subsection, we describe the symmetric multigrid algorithm and prove its convergence. We consider the sequence of discretely divergence free spaces

$$Z_0, Z_1, \dots, Z_J.$$

We define the symmetric positive definite bilinear forms $a_j(.,.)$ and $(.,.)_j$ on $Z_j \times Z_j$ for $j = 0,1,\cdots,J$ by

$$a_j(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{v}dx \quad \forall\mathbf{u},\mathbf{v} \in Z_j,$$

and

$$(\mathbf{u},\mathbf{v})_j = \int_\Omega \mathbf{u}.\mathbf{v}dx \quad \forall\mathbf{u},\mathbf{v} \in Z_j.$$

The norm corresponding to $(.,.)_j$ will be denoted by $\|.\|_j$. We shall develop multigrid algorithms for the solution of the following problem: Given $f \in Z_J$, find $\mathbf{u} \in Z_J$ such that

$$a_J(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_J \qquad \forall, \mathbf{v}, \in \ Z_J.$$

For $j = 0,1,\dots,J$. let $A_j: Z_j \to Z_j$ be the discretization operator on level j given by

$$(A_j\mathbf{v},\mathbf{w})_j = a_j(\mathbf{v},\mathbf{w}) \qquad \forall, \mathbf{w}, \in \ Z_j.$$

The operator A_j is clearly symmetric in both $a_j(.,.)$ and $(.,.)_j$ inner products and positive definite. Also, we define the operators $p_{j-1}: Z_j \to Z_{j-1}$ and $p^0_{j-1}: Z_j \to Z_{j-1}$ by

$$a_{j-1}(p_{j-1}\mathbf{v},\mathbf{w}) = a_j(\mathbf{v}, I_j\mathbf{w}), \quad \forall\, \mathbf{w} \in \ Z_{j-1},$$

and

$$(p^0_{j-1}\mathbf{v},\mathbf{w})_{j-1} = (\mathbf{v}, I_j\mathbf{w})_j, \quad \forall\, \mathbf{w} \in \ Z_{j-1},$$

where $I_j: Z_{j-1} \to Z_j$ for $j = 1,2,\dots,J$ is a prolongation operator. Note that I_jP_{j-1} is symmetric in the $a_j(.,.)$ inner product. Also, we require a linear smoothing operator $S_j: Z_j \to Z_j$ for $j = 1,2,\dots,J$. We assume that S_j is symmetric in the $(.,.)_j$-inner product and set $K_j = I - S_jA_j$. We further assume that K_j is nonnegative in the sense that $a_j(K_j\mathbf{u},\mathbf{u}) \geq 0$ for all $\mathbf{u} \in Z_j$.

The convergence rate for the multigrid algorithm on the jth level is measured by a convergence factor $\delta_j < 1$ satisfying

$$|a_j((I - B_jA_j)\mathbf{v},\mathbf{v})| \leq \delta_j a_j(\mathbf{v},\mathbf{v}) \qquad (9)$$

for all $\mathbf{v} \in Z_j$, where the multigrid operator $B_j: Z_j \to Z_j$ is defined by induction and is given as follows([2]).

Multigrid Algorithm

Set $B_0 = (A_0)^{-1}$. Assume that B_{j-1} has been defined and define B_jb for $b \in Z_j$ as follows:

(1) Set $x^0 = 0$ and $q^0 = 0$.

(2) Define x^l for $l = 1,\cdots,m(j)$ by

$$x^l = x^{l-1} + S_j^{(l+m(j))}(b - A_jx^{l-1}).$$

(3) Define $y^{m(j)} = x^{m(j)} + I_jq^p$ where q^i for $i = 1,\cdots,p$ is defined by

$$q^i = q^{i-1} + B_{j-1}[P^0_{j-1}(b - A_jx^{m(j)}) - A_{j-1}q^{i-1}].$$

(4) Define y^l for $l = m(j)+1,\cdots,2m(j)$ by

$$y^l = y^{l-1} + S_j^{(l+m(j))}(b - A_jy^{l-1}).$$

(5) Set $B_jb = y^{2m(j)}$.

In this algorithm, $m(j)$ is a positive integer which may vary from level to level and determines the number of pre and post smoothing iterations. If $p = 1$, we have a v-cycle multigrid algorithm. If $p =$

2, we have a w-cycle algorithm. A variable v-cycle algorithm is one in which the number of smoothing $m(j)$ increases exponentially as j decreases (i.e., $p = 1$ and $m(j) = 2^{J-j}$). The above multigrid is called symmetric multigrid algorithm. To estimate the convergence of multigrid algorithm, we need some conditions concerning smoothing and stability. Based on the methodology developed in [2], two very important ingredients in convergence analysis of non-nested multigrid methods are the following conditions:

Condition(A1): There is a constant C_S which does not depend on j such that the smoothing procedure satisfies

$$\frac{\| u \|_j^2}{\lambda_j} \leq C_S(S_j\mathbf{u},\mathbf{u})_j \quad \forall \mathbf{u} \in Z_j.$$

We note that when S_j is the Richardson smoothing iteration defined by $S_j = \omega\lambda_j^{-1}$ where $0 < \omega < 2$ and λ_j the maximum eigenvalue of A_j, then Condition A1 holds in equality with $C_S = 1/\omega$.

Condition(A2):

$a_j(I_j\mathbf{v}, I_j\mathbf{v}) \leq a_{j-1}(\mathbf{v},\mathbf{v}) \quad \forall \mathbf{v} \in Z_{j-1}$,
for $j = 1, \cdots, J$.

Now, we want to show Condition A2 is satisfied for our discretization.

Lemma 3 If a uniform mesh is used on polygonal domain Ω, then the Condition A2 is satisfied.

Proof: For $\mathbf{v} \in Z_{j-1}$,

$$\begin{aligned} a_{j-1}(\mathbf{v},\mathbf{v}) &= \sum_{T\in\Gamma_{j-1}} \int_T |\nabla\mathbf{v}|^2 dx \\ &= \sum_{E\in\Gamma_j} \int_E |\nabla\mathbf{v}|^2 dx \end{aligned} \tag{10}$$

Since $\mathbf{v} \in P2$, then $|\nabla\mathbf{v}|^2$ is a quadratic polynomial and therefore

$$\int_E |\nabla\mathbf{v}|^2 dx = \frac{A(E)}{3}\sum_{i=1}^{3} |\nabla\mathbf{v}(M_i)|^2,$$

where $A(E)$ is the area of triangle E and M_i for i=1,2,3 are the midpoint of edges of E. On the other hand, let E_1 and E_2 be two neighbor triangles in Γ_j with p_1, p_2, p_3 and p_4 as vertices of parallelogram $E_1 \cup E_2$ where p_1p_3 and p_2p_4 are diagonals (see Fig 9). Now, by Taylor expansion, we have:

$$\begin{cases} \mathbf{v}(p_1) - \mathbf{v}(p_3) = \big(x(p_1) - x(p_3)\big)\mathbf{v}_x(M) \\ \qquad\qquad +\big(y(p_1) - y(p_3)\big)\mathbf{v}_y(M) \\ \mathbf{v}(p_2) - \mathbf{v}(p_4) = \big(x(p_2) - x(p_4)\big)\mathbf{v}_x(M) \\ \qquad\qquad +(y(p_2) - y(p_4))\mathbf{v}_y(M). \end{cases} \tag{11}$$

Regarding the relation (11) and the fact that in an arbitrary paralellogram

$d = (x(p_1) - x(p_3)).(y(p_2) - y(p_4)) - (y(p_1) - y(p_3)).(x(p_2) - x(p_4)) \neq 0$

we have:

$$\begin{aligned} |\nabla\mathbf{v}(M)|^2 = |[(&\mathbf{v}(p_1) - \mathbf{v}(p_3))(y(p_2) \\ &-y(p_4)) - (\mathbf{v}(p_2) \\ &-\mathbf{v}(p_4))(y(p_1) - y(p_3))]/d|^2 \\ &+|[(\mathbf{v}(p_2) - \mathbf{v}(p_4))(x(p_1) \\ &-x(p_3)) - (\mathbf{v}(p_1) \\ &-\mathbf{v}(p_3))(x(p_2) - \\ x(p_4))]/d|^2, \end{aligned} \tag{12}$$

where M is the midpoint of joint edges of E_1 and E_2. Now, by (10) and (12) for $v \in Z_{j-1}$

$$a_{j-1}(\mathbf{v},\mathbf{v}) = \sum_{E\in\Gamma_j} \frac{A(E)}{3}\sum_{i=1}^{3} |\nabla\mathbf{v}(M_i)|^2$$

$$\begin{aligned} = \sum_{E\in\Gamma_j} \frac{A(E)}{3}\sum_{i=1}^{3} \\ &|[(\mathbf{v}(p_{1_i}) - \mathbf{v}(p_{3_i}))(y(p_{2_i}) - y(p_{4_i})) \\ &-(\mathbf{v}(p_{2_i}) - \mathbf{v}(p_{4_i}))(y(p_{1_i}) - y(p_{3_i}))]/d_i|^2 \\ &+|[(\mathbf{v}(p_{2_i}) - \mathbf{v}(p_{4_i}))(x(p_{1_i}) - x(p_{3_i})) \\ &-(\mathbf{v}(p_{1_i}) - \mathbf{v}(p_{3_i}))(x(p_{2_i}) - x(p_{4_i}))]/d_i|^2, \end{aligned}$$

where p_{1_i}, p_{2_i} and p_{3_i} are the vertices of triangle E and p_{4_i} is the third vertex of a neighbor triangle E' so that $p_{2_i}p_{4_i}$ forms the diagonal of paralellogram $E \cup E'$ and also

$d_i = (x(p_{1_i}) - x(p_{3_i})).(y(p_{2_i}) - y(p_{4_i})) - (y(p_{1_i}) - y(p_{3_i})).(x(p_{2_i}) - x(p_{4_i}))$.

A similar result holds for every $v \in Z_j$.

Therefore, the condition (A2) easily follows from the difinition of I_j in Section (3.1).

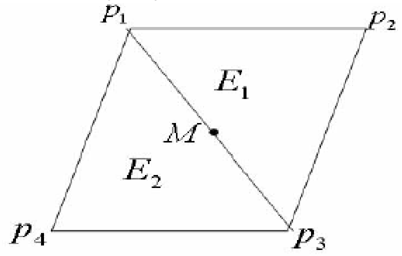

Figure 9. A sample parallelogram.

Now, the following theorem guarantees the convergence of multigrid algorithm with new intergrid transfer operator given in Section 3.1.

Theorem 4 Assume that (A.1) and (A.2) hold. Then (9) holds for some $\delta_j < 1$.

Proof: See [2].

4 Numerical experiments

We consider the Stokes problem (1) and assume that $\Omega = [0,1] \times [0,1]$ is a unit square. Let $g = 2x^2y^2(1-x)^2(1-y)^2$ and choose the vector $\mathbf{f}$ such that $\mathbf{u} = \mathbf{curl}(g)$ be the exact solution of (1). Then, $\mathbf{u}$ holds in the boundary and incompressibility conditions. We have partitioned Ω to a triangular grid as usual and the sequence of triangulation is obtained from coarsest level by regular subdivision. The coarsest level partitioned to three parts in x and y directions (see Figure 10).

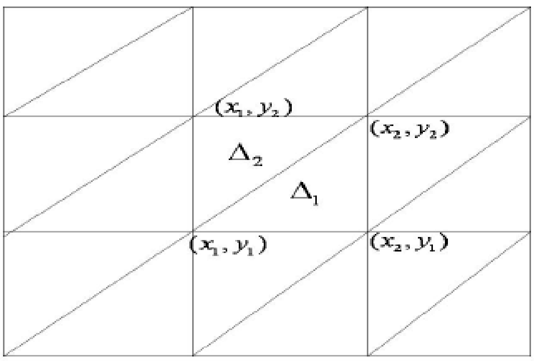

Figure 10. Coarsest level with 3 partitions in x and y directions.

For simplicity in the computation of $\int_\Delta \mathbf{f}.\mathbf{v}dx$ in which $\mathbf{v} \in Z_j$, we can use two change of variables. According to Figure 10, we have used the change of variables $x = x_1 + (x_2 - x_1)(1+\alpha)/2$ and $y = y_1 + (y_2 - y_1).(1-\alpha)(1+\beta)/4$ for triangle Δ_1 and $x = x_1 + (x_2 - x_1)(1-\alpha)(1+\beta)/4$ and $y = y_1 + (y_2 - y_1)(1+\alpha)/2$ for triangle Δ_2. Thus, by

$$\int_\Delta \mathbf{f}.\mathbf{v}d\mathbf{x} = \int_D J(\alpha,\beta)g(\alpha,\beta)d\alpha d\beta \quad (13)$$

where $J(\alpha,\beta) = \frac{\partial(\alpha,\beta)}{\partial(x,y)}$ is the Jacobian function, the integration on a triangle Δ is converted to a square $D = [-1,1] \times [-1,1]$. Now, by twice applying Gauss integration formula on (13), it is easily computed. We have solved this problem by Matlab R2023b software. Since in the higher levels (higher than 3), the matrix A (which it is obtained by finite element method) can not be stored in the computer, We have stored the matrix A by some block matrices the number of which increases with an increase in the levels. Therefore, there is no explicit matrix A and just some block matrices exist that their combinations would create the matrix A. Also, the Richardson iteration method

$$x^{k+1} = x^k + \omega_j(b_j - A_jx^k)$$

for solving the system of $A_jx = b_j$ is used where the smoothing operator

$$S_j = \omega_j = \frac{2}{\| A_j \|_\infty}.$$

Of course, if we assume that λ_{max}^j be the maximum eigenvalues of the stiffness matrix A_j in the level j, then with $\omega_j = 2/\lambda_{max}^j$, the Richardson iteration would be converged faster than $\omega_j = 2/\| A_j \|_\infty$. But, there are no explicit stiffness matrices and hence we can not drive the eigenvalues of A_j . However, since $\lambda_{max}^j \le \| A_j \|_\infty$, then $2/\| A_j \|_\infty \le 2/\lambda_{max}^j$. Hence, this option of smoothing operator is suitable. We have considered Problem (1) with number of iterations

$m(j) = 2$ in pre and post smoothing processes. Figures 11 and 12 illustrate the approximated velocities (for a v-cycle algorithm) and exact velocities in the level 3, respectively. We observe that the approximated solution is in good agreement with the exact solution. Let also $\mathbf{u}_{i,j}$ and $\mathbf{v}_{i,j}$ be the value of exact and approximated solutions in the point (x_i, y_j), respectively. Also, we let $\mathbf{e}$ be a vector with components $\mathbf{u}_{p,q} - \mathbf{v}_{p,q}$ and $\| \mathbf{e} \|_j$, the ℓ_2 norm of the error $\mathbf{e}$ in the level j. We have provided the error $\| \mathbf{e} \|_j$ from level 2 to level 8 for v-cycle, w-cycle and variable v-cycle (vv-cycle) algorithms. Table 1 shows $\| \mathbf{e} \|_j$ and as we observe, when the level j increases, the ℓ_2 norm of error would be decreased.

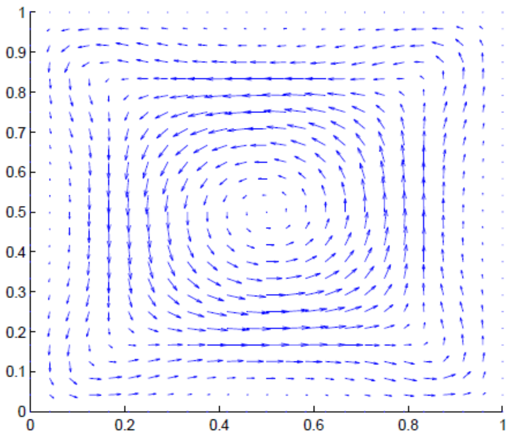

Figure 11. The approximated values of velocities in the level 3 for Problem (1).

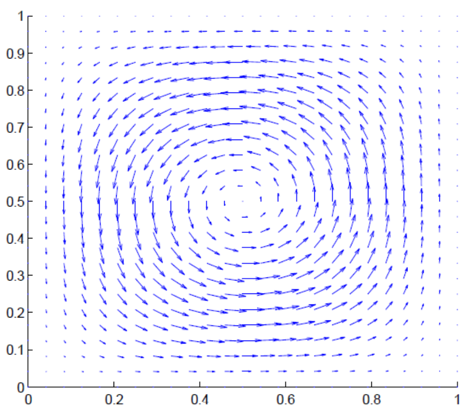

Figure 12. The exact values of velocities in the level 3 for Problem (1).

Table 1: Computation of $\| \mathbf{e} \|_j$ by MG algorithm.

$Level$	v $cycle$	w $cycle$	vv $cycle$
2	2.5e-2	2.5e-2	2.6e-2
3	1.3e-2	1.3e-2	1.3e-2
4	6.48e-3	6.47e-3	6.48e-3
5	3.24e-4	3.20e-4	3.26e-4
6	1.02e-4	1.01e-4	1.06e-4
7	7.56e-5	7.42e-5	8.01e-5
8	4.32e-5	4.28e-5	4.65e-5

5 Summary and Conclusion

In this paper, we have presented an optimal multilevel preconditioner for the divergence-free part of a P2-P1 discretization of the two dimensional Stokes problem which contains a new prolongation operator preserving the discrete divergence-free property. The convergence of mulrigrid algorithm has been given for uniform meshes. We do not know the convergence of multigrid algorithm with intergrid transfer operators given in Section 3 for non-uniform meshes and generalized Stokes problem. AT would like to thank Professor Peter Oswald from Jacobs University, Bremen, Germany for his useful discussion and encouragement.

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