Hebron University

Faculty of Science and Technology

Mathematics Department

# Cauchy integral formula and Residues

Prepared by:

Muhammad Ameen Abuawad

Supervisor:

Dr. Ali Altwaiha

Second Semester

2013

## Dedication

To those who taught us letters of gold and formulated their thoughts

as Lighthouse to light up our way of science and success,

our teachers

To the origin of love my parents

To those who travelled together as we make the road to success,

my colleagues.

To those who walk besides me to hold me when I am down,

my friends

# Table of content

# §1: introduction

## §1.1: Historical introduction to complex analysis:

Complex analysis is splendid realm within the world of mathematics, unmatched for its beauty and power. It has sometimes unexpected application to virtually every part of mathematics. It is broadly applicable beyond mathematics, and in particular, it provides powerful tools for the science and engineering.

The quadratic equation $x^2 + 1 = 0$ has no solution in real-number system because there is no real number whose square is -1. New types of numbers, called complex numbers, have been introduced to provide solution to such equations.

As early as the 16th century, a symbol $\sqrt{-1}$ was introduced to provide solution of the quadratic equation $x^2+1=0$ this symbol, later denoted by the litter $\boldsymbol{i}$ , was regarded as a fictitious or imaginary numbers which could be manipulated algebraically like an ordinary real number, except that its square was -1.

Later in the 18th century, Karl Gauss (1777-1855) and William Hamilton (1805-1865) independently and almost simultaneously proposed the idea of defining complex numbers as the order pairs (a ,b) of real numbers endowed with certain special properties. In the same time, Euler discovered the connection between trigonometric function and exponential functions through complex analysis and he invented the notation $\boldsymbol{e^{i\theta}}$. However, it was not until the 19th century that the foundations of complex analysis were laid. Among the many mathematicians and scientists who contributed, there are three who stand out as having influenced decisively the course of development of complex analysis. The first is Augustin-Louis Cauchy (1789-1857), who development the theory systematically, with the complex integral calculus, Cauchy's theorem, and the Cauchy integral formula playing fundamental roles.

The other two are Karl Theodor Weierstrass (1815-1897) and Bernhard Riemann (1826-1866), who appeared on the mathematical scene about the middle of the 19th century. Weierstrass developed the theory from a starting point of convergent power series, and this approach led towards more formal algebraic developments. Riemann contributed a more geometric point of view. His ideas had a tremendous impact not only on complex analysis but also upon mathematics as a whole, through his view took hold only gradually.

One of the most important subject in the complex is the integration, so that the scientists give the integration some focus to study and make it applicable. The most scientist who provided accomplishments in this field is Cauchy, who is provide his integration formula as alternative for some method of integration like contour integral, antiderivatives, and Goursat theorem .

In this research, we will discuss Cauchy integral formula and one of the most important theorems that depends on Cauchy integral formula, which is residue theorem, but first we will introduce the integration in general (in real valued functions) and the integration in the complex.

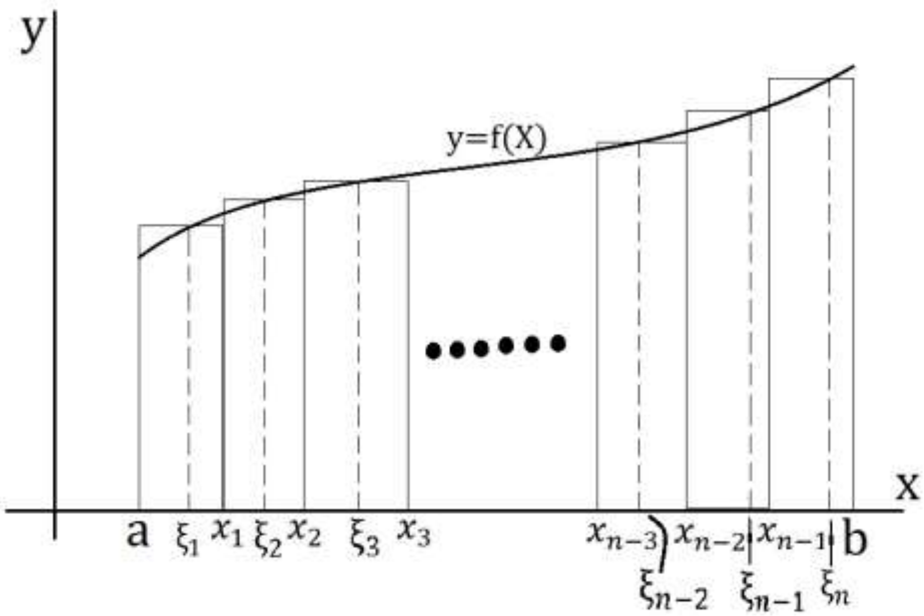
## §1.2: Introduction to integral:

The geometric problems that motivated the development of the integral calculus (determination of lengths, areas, and volumes) arose in the ancient Egyptian civilization. Where solutions were found, the related to concrete problems such as the measurement of a quantity of grain. Greek philosophers took a more abstract approach. In fact, Eudoxus (around 400 B.C.) and Archimedes (250 B.C.) formulated ideas of integration, as we know it today.

Integral calculus developed independently, and without an obvious connection to differential calculus, the calculus became a 'whole' in the last part of the 17$^{th}$ century when Isaac Barrow, Isaac Newton, and Gottfried Leibniz (with help from others) discovered that the integral of a function could be found by asking what was we differentiated to obtain that function .

The following introduction of integration is the usual one. It displays the concept geometrically and then defines the integral in 19$^{th}$ century language of limits. This form of definition establishes the basis for a wide variety of applications.

Consider the area of the region bounded by $y = f(x)$, the x-axis, and the joining vertical segments (coordinates) x=a & x=b. as shown in figure below.




Subdivide the interval $a \le x \le b$ into **n** sub-interval by means of the points $x_1, x_2, \ldots \ldots \ldots \ldots, x_{n-1}$ chosen arbitrarily. In each of the new intervals, $(a, x_1), (x_1, x_2), \ldots \ldots, (x_{n-1}, b)$ choose points $\xi_1, \xi_2, \ldots \ldots, \xi_{n-1}$ arbitrarily.

Form the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \ldots\ldots\ldots\ldots + f(\xi_n)(b - x_{n-1}) \qquad \text{(i)}$$

By writing $x_0 = a$, $x_n = b$, and $x_k - x_{k-1} = \Delta x_k$, this can be written as:

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k \qquad \text{(ii)}$$

Geometrically, this sum represents the total area of all rectangles in the above figure.

We now let the number of subdivisions $\boldsymbol{n}$ increase in such a way that each $\Delta x_k \to 0$. If as a result the sum (i) or (ii) approaches a limit which does not depend on the mode of subdivision, we denote this limit by:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \int_a^b f(x)dx$$

This is called the definite integral of $f(x)$between a and b. This symbol $f(x)dx$ is called the integrand, and [a,b] is called the range of integration. We call a and b the limit of integration, a being the lower limit of integration and b the upper limit.

In complex variable the situation is change a bit, the integration was derived by differentiation as follows.

Let $f(z) = w(t) = u(t) + iv(t)$ ……………..(iii)

where the function $u$ and $v$ are real-valued functions of $t$.

The derivative $w'(t)$, or $\frac{d(w(t))}{dt}$, of the function (iii) at a point $t$ is defined as :

$$w'(t) = u'(t) + iv'(t)$$

Provided each of the derivatives $u'$and $v'$ exist at $t$, so that the integration provide as :

$$\int_a^b w(t)\,dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

**Example:**

$$\int_0^1 (1+it)^2\,dt = \int_0^1 (1+t^2)dt + i\int_0^1 2t\,dt = \frac{2}{3} + i$$

# Terminology:

**Curves (contours) are** functions $\gamma: [a, b] \rightarrow \mathfrak{C}$, that is $\gamma(t) = x(t) + iy(t)$ where $x(t), y(t)$ are continuous.

**Simple closed contour:** the function $\sigma : [a, b] \rightarrow \mathfrak{C}$ is simple closed when if $a < t_1 < t_2 < b$ then $\sigma(t_1) \neq \sigma(t_2) \neq \sigma(a) = \sigma(b)$

**Complex domain** is any connected open subset of the complex plane $\mathfrak{C}$

**Simply Connected domain**: a domain $D$ in $\mathfrak{C}$ is simply connected if the inside of every simple closed curve in the domain is in the domain.

**Open Disc**: The open disc of radius $r$ and center $z_0$ is the set $\{z: |z - z_0| < r\}$

**Interior Point**: If $S$ is a set in $\mathfrak{C}$, then $z_0$ is an interior point $if$ $(\exists r > 0)$ such that an open disc of radius $r$ centered at $z_0$ is a subset of $S$.

**Complex Differentiable**: $f$ is complex differentiable at $z$ if

$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists. and it is called $f'(z)$.

**ML inequality**: If $M$ is an upper bound of the function and L is the arc length of the curve C, then $\left|\oint_C u(z)dz\right| \leq ML$

**Analytic Function**: Suppose $U$ is a connected open subset of $\mathfrak{C}$. Then $f: U \rightarrow \mathfrak{C}$ is complex analytic, if $f$ is complex differentiable at every point of $U$.

**Singularity**: If a complex function $f$ fails to be analytic at a point $z = z_0$, then this point is said to be a singularity or singular point of the function

**Order of a zero:** Suppose $f(z)$ is analytic. If $f(z_0) = 0$, then the order of the zero at $z_0$ is the smallest integer $k$ such that $f^{(k)}(z_0) \neq 0$

Isolated Singularity: a function $f(z)$ has an isolated singularity at $z_0$ iff $\exists r > 0$ such that $f(z)$ is analytic in $0 < |z - z_0| < r$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

**Removable Singularity**: a function $f(z)$ has a Removable singularity at $z_0$ if it have not any negative power terms.

**Pole**: a function $f(z)$ has a Pole at $z_0$ if it have finite negative power terms.

**Essential Singularity**: a function $f(z)$ have an Essential Singularity at $z_0$ if it have infinite negative power terms

Result from Cauchy- Goursat theorem:

$$\oint_c \frac{dz}{z - z_0} = 2\pi i$$

**Positively oriented curve** is a simple closed curve such that it is traveled on Counterclockwise.

# §2: Cauchy integral formula

## §2: Cauchy integral formula:

Integral representation formulas are powerful tools for studying functions. One application of an integral representation is to estimate the size of the function being represented. Another is to obtain formula for derivative. The prototype of the integral representation is provided by the Cauchy integral formula, which represent an analytic function.

The integral representation will allow us to show that all the derivatives of analytic function are analytic. In addition, it allow us to obtain power series expansion for analytic function.

### §2.1: First Formula:

The idea in this theorem is: If $f$ is analytic in a simply connected domain D and $z_0$is any point in D, then the quotient $\frac{f(Z)}{(z-z_0)}$is not analytic in D. As a result, the integral $\frac{f(Z)}{(z-z_0)}$ is not always zero around a simple closed contour C that contains $z_0$. This remarkable result indicates that the values of an analytic function $f$ at points inside a simple closed contour C are determined by the values of $f$ on the contour C.

**Theorem: (Cauchy integral formula)**

Let $f$ be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If $z_0$is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z - z_0}\, dz \qquad (i)$$

Proof: Let D be a simply connected domain, C a simple closed contour in D, and $z_0$ an interior point of C. In addition, let $C_1$ be a circle centered at $z_0$ with radius small enough that it is interior to C.

By contours, we can write

$$\oint_{c} \frac{f(z)}{z - z_0}\, dz = \oint_{c_1} \frac{f(z)}{z - z_0}\, dz \qquad (ii)$$

We wish to show that the value of the integral on the right in $(i)$ is $2\pi i\, f(z_0)$.

We add and subtract $f(z_0)$ to the right hand side $(ii)$.so we obtain

$$\oint_{c_1} \frac{f(z)}{z - z_0}\, dz = \oint_{c_1} \frac{f(z_0) - f(z_0) + f(z)}{z - z_0}\, dz$$

$$= f(z_0) \oint_{c_1} \frac{dz}{z - z_0} + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad (iii)$$

But we know that

$$\oint_{c_1} \frac{dz}{z - z_0} = 2\pi i$$

Thus, $(iii)$ becomes:

$$\oint_{c_1} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) + \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad (iv)$$

Since $f$ is continuous at $z_0$ for any arbitrarily small $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever$|z - z_0| < \delta$. In particular, if we choose the circle $C_1$ to be $|z - z_0| = \frac{\delta}{2} < \delta$, then by the ML-inequality the absolute value of the integral on the right side of (4) satisfies

$$\left| \oint_{c_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \le \frac{\varepsilon}{\delta/2} 2\pi \left(\frac{\delta}{2}\right) = 2\pi\varepsilon.$$

In other words, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle $C_1$ to be sufficiently small. This can happen only if the integral is zero.

By dividing both sides of the equation $(iv)$ by $2\pi i$ we have

$$\frac{1}{2\pi i}\left(\oint_{c_1} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) + 0\right)$$

$$\therefore \implies \quad f(z_0) = \frac{1}{2\pi i}\oint_{c_1} \frac{f(z)}{z - z_0}\, dz \quad \blacksquare$$

The Cauchy integral formula can be used to evaluate contour integrals.

Since we often work problems without a simply connected domain explicitly defined, a more practical restatement of the theorem is :

If $f$ is analytic at all points within and on a simple closed contour C, and $z_0$is any point interior to C, then $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz$.

**Example:2.1.1:**

Evaluate $\oint_C \frac{z^2-4z+4}{z+i} dz$ where C is the circle $|z| = 2$.

Solution:

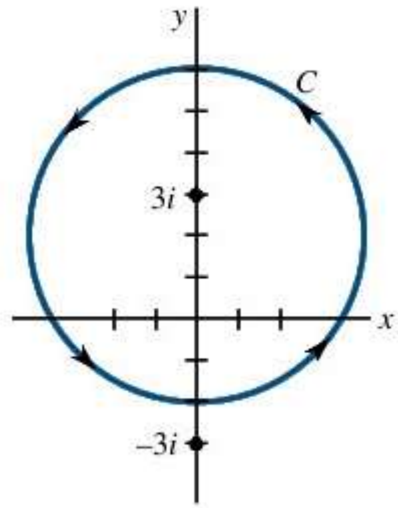
First we identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C.

Next, we observe that $f$ is analytic at all points within and on the contour C. thus by the Cauchy integral formula we obtain

$$\oint_C \frac{z^2 - 4z + 4}{z + i} dz = 2\pi i f(-i)$$

$$= 2\pi i(3 + 4i) = 2\pi(-4 + 3i).$$

**Example: 2.1.2**: Evaluate: $\oint_c \frac{z}{z^2+9}\, dz$ Where C is the circle$|z-2i|=4$.




Solution:

By factoring $z^2+9=(z-3i)(z+3i)$, we see that $3i$ is the only point within the closed contour at which the integrand fails to be analytic.

Now by writing $\frac{z}{z^2+9}=\frac{\frac{z}{z+3i}}{z-3i}$ we can identify $f(z)=\frac{z}{z+3i}$.this function is analytic at all points within and on the contour C. From the Cauchy integral formula we then have

$$\oint_c \frac{z}{z^2+9}\, dz=\oint_c \frac{\frac{z}{z+3i}}{z-3i} dz=2\pi i f(3i)=2\pi i\frac{3i}{6i}=\pi i$$

**§2.2: Second Formula:**

We can now use the Cauchy integral formula to prove that all the derivative for an analytic function is analytics; that is, if $f$ is analytic at a point $z_0$, then $f', f'', f'''$, and so on, are also analytic at $z_0$. Moreover, the values of the derivatives $f^{(n)}(z_0), n = 1, 2, 3, \dots$ , are given by a formula similar to Cauchy integral formula.

**Theorem:**

Let $f$ be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If $z_0$is any point within C, then

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_{c_1} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \qquad (i)$$

**Proof:**

Let $z_0 \in D$. We will prove that all the derivatives exist at $z_0$. First, let us prove the above formula for $n = 1$. We use Cauchy's integral formula for $f$ to evaluate

$$\frac{f(z_0+h)-f(z_0)}{h}$$

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i h}\left[\oint_c \frac{f(z)}{z-(z_0+h)}dz - \oint_c \frac{f(z)}{z-z_0}dz\right]$$

$$= \frac{1}{2\pi i h}\oint_c \frac{hf(z)}{\big(z-(z_0+h)\big)(z-z_0)}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{f(z)}{\big(z-(z_0+h)\big)(z-z_0)}dz$$

Now

$$\frac{1}{\big(z-(z_0+h)\big)(z-z_0)} = \frac{1}{(z-z_0)^2} + \frac{1}{\big(z-(z_0+h)\big)(z-z_0)} - \frac{1}{(z-z_0)^2}$$

$$= \frac{1}{(z-z_0)^2} + \frac{h}{\big(z-(z_0+h)\big)(z-z_0)^2}$$

So

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^2}dz +$$

$$\frac{1}{2\pi i}\oint_c \frac{hf(z)}{(z-(z_0+h))(z-z_0)^2}dz \qquad (i)$$

To get the result, we need to prove that the limit of $(i)$ goes to 0 as $h$ goes to 0. We are going to find an upper bound for

$$\left|\frac{f(z)}{(z-(z_0+h))(z-z_0)^2}\right|$$

First, $f$ is continuous on C as it is analytic there,so for all $z \in C$, we have $|f(z)| \leq M$ for some $M > 0$. Also, let d = min{ $|z - z_0|$ } as $z$ runs along C then for all $z \in C$, $|z - z_0| \geq d$ so

$$\left|\frac{f(z)}{(z-z_0)^2}\right| \leq \frac{M}{d^2}$$

Then, from the properties of the modulus, we always have

$|z - z_0 - h| \geq ||z - z_0| - |h||$. So let us choose $h$ such that $0 < |h| < \frac{d}{2}$. Then $|z - z_0 - h| \geq d - \frac{d}{2} = \frac{d}{2}$ so

$$\left|\frac{f(z)}{(z-(z_0+h))(z-z_0)^2}\right| \leq \frac{2M}{d^3}$$

Then

$$|(i)| = \left|\frac{1}{2\pi i}\oint_c \frac{hf(z)}{(z-(z_0+h))(z-z_0)^2}dz\right| \leq |h|\frac{ML}{\pi d^3}$$

Where $L = \oint_c |dz|$ is the length of C. so as $|h| \to 0$, we deduce that $(i) \to 0$ and therefore

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^2}dz$$

Now we proceed by induction. Suppose, for $n \geq 1$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{c_1} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Then

$$\frac{f^{(n)}(z_0 + h) - f^{(n)}(z_0)}{h} = \frac{n!}{2\pi i h} \oint_{c} f(z) \left( \frac{1}{(z - z_0 - h)^{n+1}} - \frac{1}{(z - z_0)^{n+1}} \right) dz$$

Now we write

$$\frac{1}{(z - z_0 - h)^{n+1}} - \frac{1}{(z - z_0)^{n+1}} = \frac{1}{(z - z_0)^{n+1}} \frac{1}{(1 - \frac{h}{z - z_0})^{n+1}} - \frac{1}{(z - z_0)^{n+1}}$$

$$= \frac{1}{(z - z_0)^{n+1}} \left( \frac{1 - (1 - \frac{h}{z - z_0})^{n+1}}{(1 - \frac{h}{z - z_0})^{n+1}} \right)$$

And use

$$1 - a^{n+1} = (1 - a)(1 + a + a^2 + \cdots + a^n)$$

For the numerator in the brackets to get

$$\frac{1}{(z - z_0 - h)^{n+1}} - \frac{1}{(z - z_0)^{n+1}} = \frac{h}{(z - z_0)^{n+2}} \frac{1}{(1 - \frac{h}{z - z_0})^{n+1}} \quad *$$

$$\sum_{k=0}^{n} \left(1 - \frac{h}{z - z_0}\right)^{n}$$

$$= \frac{h}{(z - z_0)^{n+2}} \frac{(z - z_0)^{n+1}}{(z - z_0 - h)^{n+1}} \sum_{k=0}^{n} \left(1 - \frac{h}{z - z_0}\right)^{n}$$

Now, using the binomial,

$$\left(1-\frac{h}{z-z_0}\right)^n=\sum_{p=0}^{n}C_n^p\,(-1)^p\left(\frac{h}{z-z_0}\right)^p=1+h\sum_{p=1}^{n}C_n^p\,(-1)^p\,\frac{h^{p-1}}{(z-z_0)^p}$$

Collecting everything, we can write the following rather long equality

$$\frac{1}{(z-z_0-h)^{n+1}}-\frac{1}{(z-z_0)^{n+1}}-\frac{h(n+1)}{(z-z_0)^{n+2}}=\frac{h(n+1)}{(z-z_0)^{n+2}}*$$

$$\frac{(z-z_0)^{n+1}-(z-z_0-h)^{n+1}}{(z-z_0-h)^{n+1}}+\frac{h^2}{(z-z_0-h)^{n+1}}*$$

$$\sum_{k=0}^{n}\sum_{p=1}^{k}C_n^p(-1)^p\,\frac{h^{p-1}}{(z-z_0)^{p+1}}$$

Now the nice thing is that the modulus of the right hand side is $|h|^2$ times some constant B say involving $d$ and higher power of $|h|$. This is seen using the same bounds as before for $|\,z-z_0|$ and $|\,z-z_0-h|$:

$$\left|\frac{1}{(z-z_0-h)^{n+1}}-\frac{1}{(z-z_0)^{n+1}}-\frac{h(n+1)}{(z-z_0)^{n+2}}\right|\leq|h|^2B$$

Therefore we can write

$$\left|\frac{f^{(n)}(z_0+h)-f^{(n)}(z_0)}{h}-\frac{(n+1)!}{2\pi i}\oint_c\frac{f(z)}{(z-z_0)^{n+2}}\,dz\right|\leq\frac{|h|Bn!}{2\pi}ML$$

Where $M$ and $L$ are as before. In the limit $h\to 0$ we obtain

$$f^{(n+1)}(z_0)=\frac{(n+1)!}{2\pi i}\oint_{c_1}\frac{f(z)}{(z-z_0)^{n+2}}\,dz \qquad \blacksquare$$

**Example2.2.1: evaluate**

$$\oint_c \frac{z+1}{z^4+4z^3}dz$$

Where C is the circle $|z| = 1$

Solution:

Inspection of the integrand shows that it is not analytic at $z = 0$ and $z = -4$, but only $z = 0$ lies within the closed contour. By writing the integrand as

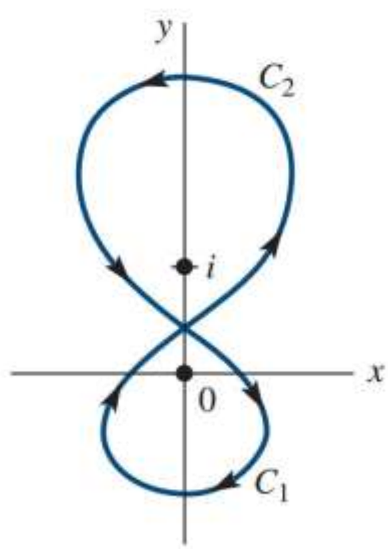
$$\frac{z+1}{z^4+4z^3} = \frac{\frac{z+1}{z+4}}{z^3}$$

We can identify $z_0 = 0, n = 2, f(z) = \frac{z+1}{z+4}$

By the quotient rule $f''(z) = \frac{-6}{(z+4)^3}$ and so we have

$$\oint_c \frac{z+1}{z^4+4z^3}dz = \frac{2\pi i}{2!}f''(0) = -\frac{3\pi}{32}i$$

**Example2.2.2**: evaluate




$$\oint_c \frac{z^3+3}{z(z-i)^2}dz$$

Where C is the contour shown in the figure

Solution:

Although C is not a simple closed contour, we can think of it as the union of two simple closed contours $C_1$ and $C_2$ as indicated in Figure above

$$\oint_{c} \frac{z^3+3}{z(z-i)^2}dz = \oint_{c_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{c_2} \frac{z^3+3}{z(z-i)^2}dz$$

$$= -\oint_{c_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}dz + \oint_{c_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = -I_1 + I_2$$

We are in a position to use both first formula and second formula. To evaluate $I_1$, we identify $z_0 = 0$ and $f(z) = (z^3 + 3)/(z - i)^2$. By first formula it follows that

$$I_1 = \oint_{c_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}dz = 2\pi i f(0) = -6\pi i$$

To evaluate $I_2$ we identify$z_0 = i, n = 1, f(z) = (z^3 + 3)/z$, and

$f'(z) = (2z^3 - 3)/z^2$. From the second formula we obtain

$$I_2 = \oint_{c_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = \frac{2\pi i}{1!}f'(i) = 2\pi i(3+2i) = 2\pi(-2+3i)$$

Finally we get

$$\oint_{c} \frac{z^3+3}{z(z-i)^2}dz = -I_1 + I_2 = 6\pi i + 2\pi(-2+3i) = 4\pi(-1+3i)$$

# §3: Residues

## §3.1: Laurent series:

The Laurent series is a representation of a complex function $f(z)$ as a series. Unlike the Taylor series which expresses $f(z)$ as a series of terms with non-negative powers of $z$, a Laurent series includes terms with negative powers. A consequence of this is that a Laurent series may be used in cases where a Taylor expansion is not possible.

If $z = z_0$is a singularity of a function $f$, then certainly f cannot be expanded in apower series with $z_0$ as its center. However, about an isolated singularity $z = z_0$, it is possible to represent f by a series involving both negative and non-negative integer powers of $z - z_0$; that is

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1(z-z_0) \quad + a_2(z-z_0)^2 + \cdots$$

we can write the above expansion as the sum of two series:

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

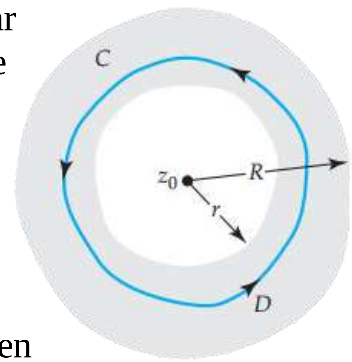
The two series on the right-hand side are given special names. The part with negative powers of $z - z_0$, i s called the principal part of the series and will converge for $\left|\frac{1}{z-z_0}\right| < r^*$ or equivalently for $|z - z_0| > \frac{1}{r^*} = r$. The part consisting of the nonnegative powers of $z - z_0$, the other part is called the analytic part of the series and will converge for $|z - z_0| < R$.

Hence, the sum of the principal part and the analytic part converges when $z$ satisfies both $|z - z_0| > r$ and $|z - z_0| < R$, that is, when z is a point in an annular domain defined by $r < |z - z_0| < R$.

By summing over negative and nonnegative integers, the series is called a Laurent series and can be written compactly as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

**Laurent's series:** Let $f$ be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then $f$ has the series representation




$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

valid for $r < |z - z_0| < R$. The coefficients $a_k$ are given by:

$$a_k = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z - z_0)^{k+1}}\, dz \ , \quad k = 0, \pm 1, \pm 2, \cdots$$

Where C is a simple closed curve that lies entirely within D and has $z_0$ in its interior. As shown in Figure above.

**§3.2: Residues:**

The residue theorem was discovered around 1814 (stated explicitly in 1831) by Cauchy as he attempted to generalize and put under one umbrella the computations of certain special integrals, some of them involving complex substitutions, that were done by Euler, Laplace, Legendre, and other mathematicians.

If we extend Cauchy's integral theorem to functions having isolated singularities, then the integral is in general not equal to zero. Instead, each singularity contributes a term called the residue. The following theorem shows that this residue depends only on the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of the function near the singularity $z_0$, since all the other powers of $z - z_0$ has single valued integrals and so integrate to zero.

Suppose that $f$ has an isolated singularity at $z_0$. We know from Laurent Theorem, that $f$ has a Laurent series in an annulus around $z_0$: for $0 < |z - z_0| < R$,

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)^1} + a_0 + a_1(z - z_0) \quad + a_2(z - z_0)^2 + \cdots$$

Furthermore, the series can be integrated term by term over any path that lies in the annulus $0 < |z - z_0| < R$. Let $C(z_0)$ be any positively oriented circle that lies in $0 < |z - z_0| < R$. If we integrate the Laurent series term by term over $C(z_0)$ and use the fact that $\oint_{C(z_0)}(z-z_0)^n\,dz\ =0$ if $n\neq -1$and $\oint_{C(z_0)}\frac{1}{z-z_0}dz\ =2\pi i$, we find $\oint_{C(z_0)}f(z)\,dz\ =a_{-1}2\pi i$; hance

$$a_{-1}=\frac{1}{2\pi i}\oint_{C(z_0)}f(z)\,dz$$

The coefficient $a_{-1}$is called the residue of $f$ at $z_0$ and is denoted by $Res\ (f,\ z_0)$ or simply $Res\ (z_0)$ when there is no risk of confusing the function $f$.

**Residue theorem**: Let C be a simple closed positively oriented path. Suppose that $f$ is analytic inside and on C, except at finitely many isolated singularities $z_1, z_2, \ldots, z_n$ inside C. Then

$$\oint_C f(z)dz=2\pi i\sum_{k=1}^{n}Res(f,z_k).$$

Proof: For every $k\ =\ 1,\ldots,n$, let $f_k(z)$ denote the principal part of $f(z)$ at $z_k$.

$f_k$ is analytic in C except at $z_k$. It follows that the function

$$g(z)=f(z)-\sum_{k=1}^{n}f_k(z)$$

Is analytic in C ,so that

$$\oint_C g(z)dz=0$$

And so

$$\oint_C f(z)dz=\sum_{k=1}^{n}\oint_C f_k(z)\ dz$$

$$\oint_C f(z)dz=2\pi i\sum_{k=1}^{n}Res(f,z_k)\qquad\blacksquare$$

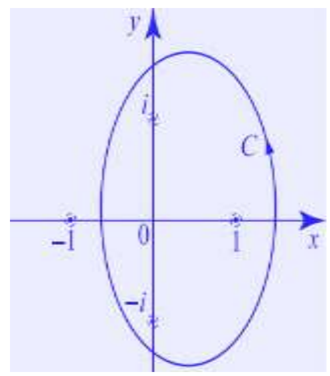
**Rule1**: Suppose that $z_0$ is an isolated singularity of $f$. Then $f$ has a simple pole at $z_0$ if and only if

$$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) \neq 0$$

**Rule2:** if $f(z) = \frac{p(z)}{q(z)}$, where $p$ and $q$ are analytic at $z_0$, $p(z_0) \neq 0$, and $q(z)$ has a simple zero at $z_0$, then

$$Res\left(\frac{p(z)}{q(z)}, z_0\right) = \frac{p(z_0)}{q'(z_0)}$$

**Example3.1:** Let C be a simple closed positively oriented path such that $1, -i$, and $i$ are in the interior of C and $-1$ is in the exterior of C .Find




$$\oint_C \frac{dz}{z^4 - 1}$$

Solution: The function $f(z) = \frac{dz}{z^4-1}$ has isolated singularities at $z = \pm 1$ and $\pm i$.

Three of these are inside C, which $\pm i$ and +1.

$$\oint_C \frac{dz}{z^4 - 1} = 2\pi i \big(Res(1) + Res(i) + Res(-i)\big)$$

We have $z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$, so $\pm 1$and $\pm i$ are simple roots of the polynomial $z^4 - 1 = 0$. Hance $f(z) = \frac{dz}{z^4-1}$ has simple poles at $z = \pm 1$ and $\pm i$.,

Using the factorization$z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$, we have at $z_0 = 1$

$$Res(1) = \lim_{z \to 1} (z-1) \frac{1}{z^4 - 1} = \lim_{z \to 1} \frac{1}{(z+1)(z-i)(z+i)}$$

$$= \frac{1}{(z+1)(z-i)(z+i)}\bigg|_{z=1} = \frac{1}{4}$$

Similarly, at $z_0 = i$ we have

$$Res(i) = \lim_{z\to i}(z-i)\frac{1}{z^4-1} = \lim_{z\to i}\frac{1}{(z-1)(z+1)(z+i)}$$

$$= \frac{1}{(z-1)(z+1)(z+i)}\bigg|_{z=i} = \frac{i}{4}$$

And at $z_0 = -i$ we have

$$Res(-i) = \lim_{z\to -i}(z+i)\frac{1}{z^4-1} = \lim_{z\to -i}\frac{1}{(z-1)(z+1)(z-i)}$$

$$= \frac{1}{(z-1)(z+1)(z-i)}\bigg|_{z=-i} = -\frac{i}{4}$$

Then

$$\oint_C \frac{dz}{z^4-1} = 2\pi i\left(\frac{1}{4}+\frac{i}{4}-\frac{i}{4}\right) = \frac{2\pi i}{2}$$

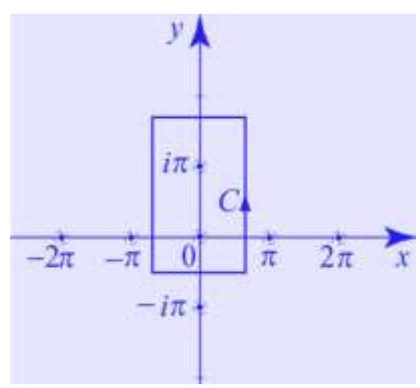
**Rule3**: Suppose that $z_0$ is a pole of order $m \geq 1$ of $f$. Then the residue of $f$ at $z_0$ is

$$Res(f, z_0) = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

Where as usual the derivative of order 0 of a function is the function itself.

**Example 3.2:** let C be the simple closed path shown in Figure, (A) Compute the residues of $f(z) = \frac{z^2}{(z^2+\pi^2)^2 \sin z}$ at all the isolated singularities inside C. (B) evaluate

$$\oint_C \frac{z^2}{(z^2+\pi^2)^2 \sin z}dz$$



Solution (A) there is three steps we will follow to answer this question.

Step1: Determine the singularities of $f$ inside C. the function $f(z) = \frac{z^2}{(z^2+\pi^2)^2 \sin z}$ is analytic except where $z^2 + \pi^2 = 0$ or $\sin z = 0$.

Thus $f$ has isolated singularities at $\pm i\pi$ and at $k\pi$ where $k$ is an integer. Only 0 and $i\pi$ are inside C.

Step2: Determine the type of the singularities of $f$ inside C. Let us start with the singularity at 0.

Using $\lim_{z\to 0} \frac{\sin z}{z} = 1$, it follows that $\lim_{z\to 0} \frac{z}{\sin z} = 1$, and so

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z}{\sin z} \frac{z}{(z^2+\pi^2)^2} = 1 \times 0 = 0$$

Then $f(z)$ has a removable singularity at $z_0 = 0$. To treat the singularities at $i\pi$, we consider $\frac{1}{f(z)} = \frac{(z+i\pi)^2(z-i\pi)^2}{z^2}$. Clearly $\frac{1}{f(z)}$ has a zero of order 2 at $i\pi$, and so $f(z)$ has a pole of order 2 at $i\pi$.

Step3: Determine the residues of $f$ inside C. At 0, $f$ has a removable singularity, so $a_{-1} = 0$, and hence the residue of $f$ at 0 is 0.

At $i\pi$, we apply Rule3, with $m = 2$ , $z_0 = i\pi$. Then

$$Res(i\pi) = \lim_{z\to i\pi} \frac{d}{dz}[(z - i\pi)^2 f(z)]$$

$$= \lim_{z\to i\pi} \frac{d}{dz}\left[\frac{z^2}{(z^2+\pi^2)^2 \sin z}\right]$$

$$= \lim_{z\to i\pi} \frac{2z(z+i\pi)\sin z - z^2((z+i\pi)\cos z + 2\sin z)}{(z+i\pi)^3 sin^2 z}$$

$$= \frac{2\sinh\pi + (-\pi\cosh\pi - \sinh\pi)}{-4\pi\, sinh^2\,\pi} = -\frac{1}{4\pi\sinh\pi} + \frac{\cosh\pi}{4\pi\, sinh^2\,\pi}$$

(B) using the result in (A) we obtain

$$\oint_C \frac{z^2}{\left(z^2+\pi^2\right)^2 \sin z} dz = 2\pi i(Res(0) + Res(i\pi))$$

$$= i\left(-\frac{1}{2\sinh \pi} + \frac{\cosh \pi}{2\, sinh^2\, \pi}\right)$$

**Rule4**: Suppose that 0 is an isolated singularity of an even function $f$. Then $Res\ (f, 0)\ =\ 0$.

**Example3.3**: Compute $Res(e^{-\frac{1}{z^2}} \cos\frac{1}{z}\ ,0)$

Solution: The function $e^{-\frac{1}{z^2}} \cos\frac{1}{z}$ is even and has an isolated (essential) singularity at 0. By Rule4, $Res\left(e^{-\frac{1}{z^2}} \cos\frac{1}{z}\ ,0\right) = 0$

**Example3.4**: evaluate $\oint_C \frac{e^z}{z^4+5z^3} dz,$ where the contour C is the circle $|z| = 2$

Solution: using the factorization $z^4 + 5z^3 = z^3(z+5)$ reveals that the integrand $f(z)$ has a pole of order 3 at $z = 0$ and a simple pole at $z = -5$ .

But only the pole $z\ =\ 0$ lies within the given contour and so we have

$$\oint_C \frac{e^z}{z^4+5z^3} dz = 2\pi i\ Res(f(z), 0) = 2\pi i \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} z^3 . \frac{e^z}{z^3(z+5)}$$

$$= \pi i \lim_{z\to 0} \frac{(z^2+8z+17)e^z}{(z+5)^3} = \frac{17\pi}{125} i$$

## References :

Gamelin, Theodore, Complex Analysis, First Edition, Los Angeles, U.S.A., Springer-Verlag, 2001.(as main book)

Lang, serge, Complex Analysis, Second Edition, New Haven, U.S.A., Springer-Verlag, 1985.

Ahlfors, Lars, Complex Analysis, Third Edition, U.S.A, McGraw-Hill, 1979.

Churchill, Ruel, complex variables and application, fifth edition, U.S.A, McGraw-Hill, 1990.

Shanahan, Patrick, A first course in complex analysis with application, U.S.A, Jones and Bartlett mathematics, 2003.