Algebraic General Topology. Book 3: Algebra. Edition 3

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Abstract. I define *space* as an element of an ordered semigroup action, that is a semigroup action conforming to a partial order. Topological spaces, uniform spaces, proximity spaces, (directed) graphs, metric spaces, etc. all are spaces. It can be further generalized to ordered precategory actions (that I call *interspaces*). I build basic general topology (continuity, limit, openness, closedness, hausdorffness, compactness, etc.) in an arbitrary space. Now general topology is an algebraic theory.

For example, my generalized continuous function are: continuous function for topological spaces, proximally continuous functions for proximity spaces, uniformly continuous functions for uniform spaces, contractions for metric spaces, discretely continuous functions for (directed) graphs.

Was a spell laid onto Earth mathematicians not to find the most important structure in general topology until 2019?

Contents

This is a draft.

It is a continuation of [1].

You can read this text without any knowledge of algebraic general topology ([1]). But to have some examples of how to apply this theory, you need to know what funcoids and reloids are and how funcoids are related with topological spaces.

CHAPTER 1

Introduction

I will show that most of the topology can be formulated in an ordered semigroup (or, more generally, an ordered precategory).

I will make this part of the book mostly self-contained, for example, reminding definitions of funcoids.

CHAPTER 2

Prerequisites

You need to know about semigroups, ordered semigroups, semigroup actions, before reading further. If in doubt, consult Wikipedia.

Filtrators are pairs of a poset and its subset (with the induced order). An important example of filtrator is the set of filters on some poset together with the subset of principal filters. (Note that I order filters *reversely* to the set inclusion relation: So for filters I have $a \sqsubseteq b \Leftrightarrow a \supseteq b$.)

I will denote meet and join on a poset correspondingly as $\sqcap$ and $\sqcup$.

I call two elements a and b *intersecting* ($a \not\asymp b$) when there is a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$. For meet-semilattices with meet operation $\sqcap$ this condition is equivalent to $a \sqcap b$ being non-least element.

I call two elements a and b *joining* ($a \equiv b$) when there is no non-greatest element c such that $c \sqsupseteq a \wedge c \sqsupseteq b$. For join-semilattices with meet operation $\sqcup$ this condition is equivalent to $a \sqcup b$ being the greatest element.

I denote $\langle f \rangle^* X = \left\{ \frac{fx}{x \in X} \right\}$.

CHAPTER 3

Basic examples

A topological space is determined by its closure operator.

Consider the semigroup formed by composing together any finite number of topological closure operators (on some fixed "universal" set).

This semigroup can be considered as its own action.

So every topological space is an element of this semigroup that is associated with an action.

The set, on which these actions act, is the set of subsets of our universal set. The set of subsets of a set is a partially ordered set.

So we have topological space defined by actions of an ordered semigroup.

Below I will define a *space* as an *ordered semigroup action element*.

This includes topological spaces, uniform spaces, proximity spaces, (directed) graphs, metric spaces, semigroups of operators, etc.

Moreover we can consider the semigroup of all functions $\mathscr{P}\mho \to \mathscr{P}\mho$ for some set $\mho$ (the set of "points" of our space). Above we showed that topological spaces correspond to elements of this semigroup. Functions on $\mho$ also can be considered as elements of this semigroup (replace every function with its "image of a set" function). Then we have an ordered semigroup action containing both topospaces and functions. As it was considered above, we can describe a function f being continuous from a space μ to a space ν by the formula $f \circ \mu \sqsubseteq \nu \circ f$. See, it's an instance of *algebraic* general topology: a topological concept was described by an algebraic formula, without any quantifiers.

CHAPTER 4

Precategories

DEFINITION 2111. A *precategory* is a directed multigraph together with a partial binary operation $\circ$ on the set $\mathcal{M}$ of edges (called the set of *morphisms* in the context of precategories) such that $g \circ f$ is defined iff $\operatorname{Dst} f = \operatorname{Src} g$ (for every morphisms f and g) such that

1°. $\operatorname{Src}(g \circ f) = \operatorname{Src} f$ and $\operatorname{Dst}(g \circ f) = \operatorname{Dst} g$ whenever the composition $g \circ f$ of morphisms f and g is defined.
2°. $(h \circ g) \circ f = h \circ (g \circ f)$ whenever compositions in this equation are defined.

DEFINITION 2112. A *prefunctor* is a pair of a function from the set of objects of one precategory to the set of objects of another precategory and a function from the set of morphisms of one precategory to the set of morphisms of that another precategory (these two functions are denoted by the same letter such as ϕ) conforming to the axioms:

1°. $\phi(f) : \phi(\operatorname{Src} f) \to \phi(\operatorname{Dst} f)$ for every morphism f of the first precategory;
2°. $\phi(g \circ f) = \phi(g) \circ \phi(f)$ for every composable morphisms f, g of the first precategory.

NOTE 2113. A semigroup is essentially a special case of a precategory (with only one object) and semigroup homomorphism is a prefunctor.

CHAPTER 5

Ordered precategories

DEFINITION 2114. *Ordered precategory* (or *poprecategory*) is a precategory together with an order on the set of morphisms conforming to the equality:

$$x_0 \sqsubseteq x_1 \wedge y_0 \sqsubseteq y_1 \Rightarrow y_0 \circ x_0 \sqsubseteq y_1 \circ x_1.$$

CHAPTER 6

Ordered semigroups

DEFINITION 2115. *Ordered semigroup* (or *posemigroup*) is a set together with binary operation $\circ$ and binary relation $\sqsubseteq$ on it, conforming both to semigroup axioms and partial order axioms and:

$$x_0 \sqsubseteq x_1 \wedge y_0 \sqsubseteq y_1 \Rightarrow y_0 \circ x_0 \sqsubseteq y_1 \circ x_1.$$

Essentially, a posemigroup is just an ordered precategory with just one object.

In this book I will call elements of an ordered semigroup *spaces*, because they generalize such things as topological spaces, (quasi)proximity spaces, (quasi)uniform spaces, (directed) graphs, (quasi)metric spaces.

As I shown above, functions (and more generally binary relations) can also be considered as spaces.

CHAPTER 7

Precategory actions

DEFINITION 2116. *Precategory action* is a prefunctor from a precategory to the category **Set**.

CHAPTER 8

Ordered precategory actions

The category **Pos** is the category whose objects are (small) posets and whose morphisms are order homomorphisms.

DEFINITION 2117. *Semiordered precategory action* on a is a precategory action $\langle\rangle$ to the category **Pos** of all partially ordered sets, such that

1°. $a \sqsubseteq b \Rightarrow \langle a\rangle x \sqsubseteq \langle b\rangle x$ for all $a, b \in S$, $x \in \mathfrak{A}$.

I call morphisms of such a precategory as *semi-interspaces.*[1]

DEFINITION 2118. *Ordered precategory action* on a is a precategory action $\langle\rangle$ to the category **Pos** of all partially ordered sets, such that

1°. $a \sqsubseteq b \Rightarrow \langle a\rangle x \sqsubseteq \langle b\rangle x$ for all $a, b \in S$, $x \in \mathfrak{A}$;

[1]The prefix inter- is supposed to mean that the morphisms may have the source different that the destination.

2°. $x \sqsubseteq y \Rightarrow \langle a \rangle x \sqsubseteq \langle a \rangle y$ for all $a \in S$, $x, y \in \mathfrak{A}$.

In other words, an ordered precategory action is a (not necessarily strictly) increasing precategory action (we consider transformations of this action to be ordered pointwise, that is by the product order).

I call morphisms of such a precategory as *interspaces*.

Note that this "inducting" is an ordered semigroup homomorphism.

CHAPTER 9

Ordered semigroup actions

DEFINITION 2119. *Curried semiordered semigroup action* on a poset $\mathfrak{A}$ for an ordered semigroup S is a function $\langle\rangle : S \to (\mathfrak{A} \to \mathfrak{A})$ such that

1°. $\langle b \circ a \rangle x = \langle b \rangle \langle a \rangle x$ for all $a, b \in S$, $x \in \mathfrak{A}$; $x, y \in \mathfrak{A}$;
2°. $a \sqsubseteq b \Rightarrow \langle a \rangle x \sqsubseteq \langle b \rangle x$ for all $a, b \in S$, $x \in \mathfrak{A}$.

I call elements of such an action *semispaces*.

DEFINITION 2120. *Curried ordered semigroup action* on a poset $\mathfrak{A}$ for an ordered semigroup S is a function $\langle\rangle : S \to (\mathfrak{A} \to \mathfrak{A})$ such that

1°. $\langle b \circ a \rangle x = \langle b \rangle \langle a \rangle x$ for all $a, b \in S$, $x \in \mathfrak{A}$; $x, y \in \mathfrak{A}$;
2°. $a \sqsubseteq b \Rightarrow \langle a \rangle x \sqsubseteq \langle b \rangle x$ for all $a, b \in S$, $x \in \mathfrak{A}$;
3°. $x \sqsubseteq y \Rightarrow \langle a \rangle x \sqsubseteq \langle a \rangle y$ for all $a \in S$.

I call elements of such an action *spaces*.

REMARK 2121. Google search for “"ordered semigroup action"” showed nothing. Was a spell laid onto Earth mathematicians not to find the most important structure in general topology?

Essentially, an ordered semigroup action is an ordered precategory action with just one object.

We can order actions componentwise. Then the above axioms simplify to:

1°. $\langle b \circ a \rangle = \langle b \rangle \circ \langle a \rangle$ for all $a, b \in S$;
2°. $\langle\rangle$ is a (not necessarily strictly) increasing;
3°. $\langle a \rangle$ is a (not necessarily strictly) increasing, for every space a.

DEFINITION 2122. A *functional ordered precategory action* is such an ordered precategory action that $\langle a \rangle = a$ for every space a.

THEOREM 2123. Each ordered precategory action induces as functional ordered precategory action, whose morphisms are the same a of the original one but with objects being posets, spaces are the actions of the original precategory, the composition operation is function composition, and order of spaces is the product order.

PROOF. That it's a precategory is obvious. The partial order is the same as the original. It remains to prove the remaining axioms.

For our precategory

$$\langle b \circ a \rangle = b \circ a = \langle b \rangle \circ \langle a \rangle.$$

$\langle\rangle$ is increasing because it's the identity function.

$\langle a \rangle$ is the same as one of the original ordered precategory action and thus is increasing. □

Having a ordered precategory action and a homomorphism to its ordered precategory, we can define in an obvious way a new ordered precategory action. The following is an example of this construction (here $(\mathsf{RLD})_{\mathrm{in}}$ is a functor of ordered precategories).

Funcoids form an ordered precategory with action $\langle\rangle$. *Reloids* form an ordered precategory with action $a \mapsto \langle (\mathsf{RLD})_{\mathrm{in}} a \rangle$. As we know from the above, funcoids are a generalization of topological spaces, proximity spaces, and directed graphs ("discrete spaces"), reloids is a generalization of uniform spaces and directed graphs. Funcoid is determined by its action. So most of the customary general topology can be described in terms of ordered precategory actions (or ordered semigroup actions, see below).

Remember that elements of our posets of objects may be such things as sets or more generally filters, they may be not just points. So our topological construction is "pointfree" (we may consider sets or filters, not points).

This part of the book is mainly about this topic: describing general topology in terms of ordered precategory actions. Above are the new axioms for general topology. No topological spaces here.

Semiordered precategory action is *ordered by elements* when

$$a \sqsubseteq b \Leftarrow \langle a \rangle \sqsubseteq \langle b \rangle$$

that is when

$$a \sqsubseteq b \Leftarrow \forall x : \langle a \rangle x \sqsubseteq \langle b \rangle x.$$

Obviously, in this case $\langle\rangle$ is a faithful functor. So our ordered precategory action is *essentially functional* (functional, up to a faithful functor).

CHAPTER 10

Ordered dagger categories and ordered semigroups with involution

DEFINITION 2124. *Dagger precategory* is a precategory together with the operation $a \mapsto a^\dagger$ (called *involution* or *dagger*) such that:

1°. $a^{\dagger\dagger} = a$;
2°. $(b \circ a)^\dagger = a^\dagger \circ b^\dagger$.

For an *ordered dagger precategory* we will additionally require $a \sqsubseteq b \Rightarrow a^\dagger \sqsubseteq b^\dagger$ (and consequently $a \sqsubseteq b \Leftrightarrow a^\dagger \sqsubseteq b^\dagger$).

DEFINITION 2125. *Semigroup with involution* is a dagger precategory with just one object.

For an *ordered semigroup with involution* or *ordered dagger precategory* we will additionally require $a \sqsubseteq b \Rightarrow a^\dagger \sqsubseteq b^\dagger$ (and consequently $a \sqsubseteq b \Leftrightarrow a^\dagger \sqsubseteq b^\dagger$).

CHAPTER 11

Topological properties

Now we have a formalism to describe many topological properties (following the idea above in this book):

Continuity is described by the formulas $f \circ a \sqsubseteq a \circ f$, $f \circ a \circ f^\dagger \sqsubseteq a$, $a \sqsubseteq f^\dagger \circ a \circ f$.

Convergence of a function f from an endomorphism (space) μ to an endomorphism (space) ν at filter x to a set or filter y is described by the formula $\langle f\rangle\langle\mu\rangle x \sqsubseteq \langle\nu\rangle y$.

Generalized limit of an arbitrary interspace f (for example, of an arbitrary (possibly discontinuous) function), see [2], is described by the formula

$$\operatorname{xlim} f = \left\{ \frac{\nu \circ f \circ r}{r \in G} \right\},$$

where G is a suitable group (consider for example the group of all translations of a vector space).

Neighborhood of element x is such a y that $\langle a\rangle x \sqsubseteq y$. *Interior* of x (if it exists) if the join of all y such that x is a neighborhood of x.

An element x is closed regarding a iff $\langle a\rangle x \sqsubseteq x$. x is open iff x is closed regarding $\langle a\rangle^\dagger$.

To define compactness[1] we additionally need the structure of filtrator $(\mathfrak{A}, \mathfrak{Z})$ on our poset. Then it is space a is *directly compact* iff

$$\forall x \in \mathfrak{A} : (x \text{ is non-least} \Rightarrow \operatorname{Cor}\langle a\rangle x \text{ is non-least});$$

a is *reversely compact* iff $a^\dagger$ is directly compact; a is *compact* iff it is both directly and reversely compact.

Denote c the element of the precategory **Set** such that $\langle c\rangle = \operatorname{Cor}$, then the above can be rewritten

$$\forall x \in \mathfrak{A} : (x \text{ is non-least} \Rightarrow \langle c \circ a\rangle x \text{ is non-least});$$

what is equivalent to $1 \sqsubseteq c \circ a$.

However, we can define compactness without specifying $\mathfrak{Z}$ as we can take $\mathfrak{Z}$ to be the *center* (the set of all its complemented elements) of the poset $\mathfrak{A}$.

The same reasoning applies to Cor' in place of Cor.

It seem we cannot define *total boundness* purely in terms of ordered semigroups, because it is a

[1]That this coincides with the traditional definition of compactness of topological spaces, follows from the well known fact that a topological space is compact iff each proper filter on it has an adherent point.

property of reloids and reloid is not determined by its action.

CHAPTER 12

A relation

Every ordered precategory action $\langle\rangle$ defines a relation R: $x\ [a]\ y \Leftrightarrow y \not\asymp \langle a\rangle x$.

If $[a]^{\dagger}{=}[a]^{-1}$ for every a, we call the action $\langle\rangle$ on an dagger precategory *intersection-symmetric*. In this case our action defines a pointfree funcoid.

A space is connected iff $x \equiv y \Rightarrow x\ [a]\ y$.

We can define open and closed functions.

CHAPTER 13

Further axioms

Further possible axioms for an ordered semigroup action with binary joins:

- $\langle f \rangle (x \sqcup y) = \langle f \rangle x \sqcup \langle f \rangle y$;
- $\langle f \sqcup g \rangle x = \langle f \rangle x \sqcup \langle g \rangle x$.

FiXme: Need to generalize for a wider class of posets.

CHAPTER 14

Restricted identity transformations

Restricted identity transformation id_p, where p is an element of a poset, is the (generally, partially defined) transformation $x \mapsto x \sqcap p$.

OBVIOUS 2126. $\operatorname{id}_q \circ \operatorname{id}_p = \operatorname{id}_{p \sqcap q}$ if p and q are elements of some poset for which binary meet is defined.

PROPOSITION 2127. $p \neq q \Rightarrow \operatorname{id}_p \neq \operatorname{id}_q$.

PROOF. $\operatorname{id}_p p = p \neq q = \operatorname{id}_q q$. □

Ordered precategory action with identities is an ordered precategory S action $\langle\rangle$ together with a function $p \mapsto \operatorname{id}_p \in S$ such that

1°. $\langle \operatorname{id}_p \rangle = \operatorname{id}_p$ whenever this equality is defined;
2°. $\operatorname{id}_p \circ x \sqsubseteq x$;
3°. $x \circ \operatorname{id}_p \sqsubseteq x$.

(I abuse the notation id_p for both interspaces and for transformations; this won't lead to inconsistencies, because as proved above this mapping is faithful on restricted identities.)

OBVIOUS 2128. For every ordered precategory action with identities, the identity transformations are entirely defined on their domains.

From injectivity it follows $\mathrm{id}_{p \sqcap q} = \mathrm{id}_p \circ \mathrm{id}_q$.

Restriction of an interspace a to element x is $a|_x = a \circ \mathrm{id}_x$.

Square restriction (a generalization of restriction of a topological space, metric space, etc.) of a space a to element x is $\mathrm{id}_x \circ a \circ \mathrm{id}_x$.

CHAPTER 15

Binary product of poset elements

DEFINITION 2129. I call an ordered precategory action *correctly bounded* when the set of interspaces between two fixed objects is bounded and:

1°. $\langle\bot\rangle x = \bot$ for every poset element x;
2°. $\langle\top\rangle x = \begin{cases} \top & \text{if } x \neq \bot, \\ \bot & \text{if } x = \bot. \end{cases}$

Binary product in an ordered semigroup action having a greatest element $\top$ is defined as $p \times q = \mathrm{id}_q \circ \top \circ \mathrm{id}_p$.

THEOREM 2130. If our action is correctly bounded, then

$$\langle p \times q\rangle x = \begin{cases} q & \text{if } x \not\asymp p, \\ \bot & \text{if } x \asymp p. \end{cases}$$

PROOF.

$$\langle p \times q \rangle x = \langle \mathrm{id}_q \circ \top \circ \mathrm{id}_p \rangle x = \langle \mathrm{id}_q \rangle \langle \top \rangle \langle \mathrm{id}_p \rangle x = q \sqcap \langle \top \rangle (p \sqcap x) = \begin{cases} q & \text{if } x \not\asymp p, \\ \bot & \text{if } x \asymp p. \end{cases}$$

□

THEOREM 2131. If our action is correctly bounded, then

$$x \left[p \times q \right] y \Leftrightarrow x \not\asymp p \wedge y \not\asymp q.$$

PROOF.

$$x \left[p \times q \right] y \Leftrightarrow y \not\asymp \langle p \times q \rangle x \Leftrightarrow y \not\asymp \begin{cases} q & \text{if } x \not\asymp p, \\ \bot & \text{if } x \asymp p. \end{cases} \Leftrightarrow x \not\asymp p \wedge y \not\asymp q.$$

□

CHAPTER 16

Separable spaces

T_1-space a when $x \ \overline{R \operatorname{Cor} a} \ y$ for every $x \asymp y$.

T_2-space or *Hausdorff* is such a space f that $f^{-1} \circ f$ is T_1-separable.

T_0-space is such a space f that $f^{-1} \sqcap f$ is T_1-separable.

T_4-space is such a space f that

$$f \circ f^{-1} \circ f \circ f^{-1} \sqsubseteq f \circ f^{-1}.$$

CHAPTER 17

Distributive ordered semigroup actions

We can define (product) order of ordered precategory actions. For functional ordered precategory actions composition is defined. So we have one more "level" of ordered precategories. By the way, it can be continued indefinitely building new and new levels of such ordered precategories.

More generally we could consider ordered precategory functors (or specifically, ordered semigroup homomorphisms). Examples of such homomorphisms are $\langle\rangle$, (FCD), $(\mathsf{RLD})_{\mathrm{in}}$.

Pointfree funcoids (and consequently funcoids) are an ordered precategory action. Reloids are also an ordered precategory action.

CHAPTER 18

Complete spaces and completion of spaces

A space a is *complete* when $\langle a\rangle\bigsqcup S = \bigsqcup\langle\langle a\rangle\rangle^{*}S$ whenever both $\bigsqcup S$ and $\bigsqcup\langle\langle a\rangle\rangle^{*}S$ are defined.

DEFINITION 2132. *Completion* of an interspace is its core part (see above for a definition of core part) on the filtrator of interspace and complete interspace.

NOTE 2133. Apparently, not every space has a completion.

NOTE 2134. It is unrelated with Cachy-completion.

CHAPTER 19

Kuratowski spaces

Definition 2135. *Kuratowski space* is a complete idempotent ($a \circ a = a$) space.

Kuratowski spaces are a generalization of topological spaces.

CHAPTER 20

Metric spaces

Let us call *most general nonnegative real metrics* (MGNRM) the precategory of all extended nonnegative ($\mathbb{R}_+ \cup \{+\infty\}$) real functions (on some fixed set) of two arguments and the "composition" operation

$$(\sigma \circ \rho)(x, z) = \inf_{y \in \mho}(\rho(x, y) + \sigma(z, y))$$

and *most general nonnegative real metric* an element of this precategory.

Remark 2136. The infimum exists because it's nonnegative.

We need to prove it's an associative operation.

Proof.

$$\begin{aligned}(\tau\circ(\sigma\circ\rho))(x,z) &= \\ &\inf_{y_1\in\mho}((\sigma\circ\rho)(x,y_1)+\tau(y_1,z)) = \\ &\inf_{y_1\in\mho}(\inf_{y_0\in\mho}(\rho(x,y_0)+\sigma(y_0,y_1))+\tau(y_1,z)) = \\ &\inf_{y_0,y_1\in\mho}(\rho(x,y_0)+\sigma(y_0,y_1)+\tau(y_1,z)).\end{aligned}$$

Similarly

$$\begin{aligned}((\tau\circ\sigma)\circ\rho)(x,z) &= \\ &\inf_{y_0,y_1\in\mho}(\rho(x,y_0)+\sigma(y_0,y_1)+\tau(y_1,z)).\end{aligned}$$

Thus $\tau\circ(\sigma\circ\rho) = (\tau\circ\sigma)\circ\rho$. □

Definition 2137. We extend MGNRM to the set $\mathscr{P}\mho$ by the formula:

$$\rho(X,Y) = \inf_{x\in X, y\in Y}\rho(x,y).$$

Remark 2138. This is well-defined thanks to MGNRM being nonnegative and allowing the infinite value.

Proposition 2139.

1°. $\rho(I\cup J, Y) = \min\{\rho(I,Y), \rho(J,Y)\}$;
2°. $\rho(X, I\cup J) = \min\{\rho(X,I), \rho(Y,J)\}$.

PROOF. We'll prove the first as the second is similar:

$$\rho(I \cup J, Y) = \inf_{x \in I \cup J, y \in Y} \rho(x, y) = \min\left\{ \inf_{x \in I, y \in Y} \rho(x, y), \inf_{x \in J, y \in Y} \rho(x, y) \right\} = \min\{\rho(I, Y), \rho(J, Y)\}.$$

□

Let a be a most general metric. I denote Δ_a the funcoid determined by the formula

$$X \; [\Delta_a]^* \; Y \Leftrightarrow \rho_a(X, Y) = 0.$$

(If a is a metric, then it's the proximity induced by it.)

Let's prove it really defines a funcoid:

PROOF. Not $\emptyset \; [\Delta_a]^* \; Y$ and not $X \; [\Delta_a]^* \; \emptyset$ because

$$\rho_a(\emptyset, Y) = \rho_a(X, \emptyset) = +\infty.$$

By symmetry, it remains to prove

$$(I \cup J) \; [\Delta_a]^* \; Y \Leftrightarrow I \; [\Delta_a]^* \; Y \vee J \; [\Delta_a]^* \; Y.$$

Really,

$$\begin{aligned}(I \cup J) \left[\Delta_a\right]^* Y \Leftrightarrow \\ \rho_a(I \cup J, Y) = 0 \Leftrightarrow \\ \min\{\rho_a(I, Y), \rho_a(J, Y)\} = 0 \Leftrightarrow \\ \rho_a(I, Y) = 0 \vee \rho_a(J, Y) = 0 \Leftrightarrow \\ I \left[\Delta_a\right]^* Y \vee J \left[\Delta_a\right]^* Y.\end{aligned}$$

□

Obvious 2140.

$$\begin{aligned}X \left[\Delta_a\right]^* Y \Leftrightarrow \\ \forall \epsilon > 0 \exists x \in X, y \in Y : |\rho_a(x, y)| < \epsilon.\end{aligned}$$

Theorem 2141.

$$\langle \Delta_a \rangle X = \prod_{\epsilon > 0} \bigcup_{x \in X} B(x, \epsilon)$$

($B(x, \epsilon)$ is the open ball of the radius ϵ centered at x).

Proof.

$$\begin{aligned}Y \not\asymp \langle \Delta_a \rangle X \Leftrightarrow X \left[\Delta_a\right] Y \Leftrightarrow \\ \forall \epsilon > 0 \exists x \in X, y \in Y : \rho_a(x, y) < \epsilon.\end{aligned}$$

$$Y \not\asymp \prod_{\epsilon>0} \bigcup_{x\in X} B_a(x,\epsilon) \Leftrightarrow$$
$$\forall \epsilon > 0 : Y \not\asymp \bigcup_{x\in X} B_a(x,\epsilon) \Leftrightarrow$$
$$\forall \epsilon > 0 \exists x \in X : Y \not\asymp B_a(x,\epsilon) \Leftrightarrow$$
$$\forall \epsilon > 0 \exists x \in X, y \in Y : \rho_a(x,y) < \epsilon.$$

□

MGNRM are also interspaces: Define the order on metric spaces by the formula

$$\rho \sqsubseteq \sigma \Leftrightarrow \forall x, y : \rho(x,y) \sqsupseteq \sigma(x,y).$$

Define the action for a metric space a as the action $\langle \Delta_a \rangle$ of its induced proximity Δ_a (see above for a definition of proximity and more generally funcoid actions $\langle\rangle$) and composition of metrics ρ, σ by the formula:

$$(\sigma \circ \rho)(x,z) = \inf_{y\in\mho} (\rho(x,y) + \sigma(z,y)),$$

where $\mho$ is the set of points of our metric space.

LEMMA 2142. $\Delta_{b\circ a} = \Delta_b \circ \Delta_a$.

PROOF. Let X, Y be arbitrary sets on a metric space.

$$\begin{aligned}
&Z \not\asymp \langle \Delta_{b\circ a} \rangle X \Leftrightarrow \\
&\quad \forall \epsilon > 0 \exists x \in X, z \in Z : \\
&\quad \inf_{y \in \mho} (\rho_a(x,y) + \rho_b(y,z)) < \epsilon \Leftrightarrow \\
&\quad \forall \epsilon > 0 \exists x \in X, y \in \mho, z \in Z : \\
&\quad \rho_a(x,y) + \rho_b(y,z) < \epsilon \Leftrightarrow \\
&\quad \forall \epsilon > 0 \exists x \in X, y \in \mho, z \in Z : \\
&\qquad (\rho_a(x,y) < \epsilon \land \rho_b(y,z) < \epsilon)
\end{aligned}$$

$$\begin{aligned}
&Z \not\asymp \langle \Delta_b \circ \Delta_a \rangle X \Leftrightarrow Z \not\asymp \langle \Delta_b \rangle \langle \Delta_a \rangle X \Leftrightarrow \\
&\quad \left\langle \Delta_b^{-1} \right\rangle Z \not\asymp \langle \Delta_a \rangle X \Leftrightarrow \\
&\quad \prod_{\epsilon > 0} \bigcup_{x \in X} B_a(x,\epsilon) \not\asymp \prod_{\epsilon > 0} \bigcup_{z \in Z} B_b(z,\epsilon) \Leftrightarrow \\
&\quad \forall \epsilon > 0 : \bigcup_{x \in X} B_a(x,\epsilon) \not\asymp \bigcup_{z \in Z} B_b(z,\epsilon) \Leftrightarrow \\
&\quad \forall \epsilon > 0 \exists x \in X, z \in Z : B_a(x,\epsilon) \not\asymp B_b(z,\epsilon) \Leftrightarrow \\
&\quad \forall \epsilon > 0 \exists x \in X, z \in Z, y \in \mho : \\
&\qquad (\rho_a(x,z) < \epsilon \land \rho_b(z,y) < \epsilon).
\end{aligned}$$

So, $Z \not\asymp \langle \Delta_{b\circ a} \rangle X \Leftrightarrow Z \not\asymp \langle \Delta_b \circ \Delta_a \rangle X$. □

Let's prove it's really an ordered precategory action:

PROOF.

- It is an ordered precategory, because $\langle a\rangle x = \langle \Delta_a\rangle x \sqsubseteq \langle \Delta_a\rangle y = \langle a\rangle y$ for filters $x \sqsubseteq y$.
- $$\langle b \circ a\rangle = \langle \Delta_{b\circ a}\rangle = \langle \Delta_b \circ \Delta_a\rangle = \langle \Delta_b\rangle \circ \langle \Delta_a\rangle = \langle b\rangle \circ \langle a\rangle;$$
- $a \sqsubseteq b \Rightarrow \langle a\rangle \sqsubseteq \langle b\rangle$ is obvious;
- $x \sqsubseteq y \Rightarrow \langle a\rangle x \sqsubseteq \langle a\rangle y$ for all $a \in S$ is obvious.

□

FiXme: The above can be generalized for the values of the metric to be certain ordered additive semigroups instead of nonnegative real numbers.

1. Functions as metrics

We want to consider functions in relations with MGNRM. So we we will consider (not only functions but also) every morphism f of category **Rel** as an MGNRM by the formulas $\rho_f(x, y) = 0$ if $x \; f \; y$ and $\rho_f(x, y) = +\infty$ if not $x \; f \; y$.

THEOREM 2143. If ρ is a MGNRM and f is a binary relation composable with it, then:

1°. $(\rho \circ f)(X, Y) = \rho(Y, \langle f\rangle^* X)$;

2°. $(f \circ \rho)(X, Y) = \rho(\langle f^{-1} \rangle^* Y, X)$.

PROOF.

$$(\rho \circ f)(x, y) = \inf_t (f(X, t) + \rho(Y, y))$$

but $f(X, t) + \rho(Y, t) = +\infty$ if not $X \; [f]^* \; \{t\}$ and $f(X, t) + \rho(Y, t) = \rho(Y, t)$ if $X \; [f]^* \; \{t\}$. So

$$(\rho \circ f)(X, Y) = \inf_{t \in \left\{ \frac{t}{X \, [f]^* \, \{t\}} \right\}} \rho(Y, t) = \inf_{t \in \langle f \rangle^* X} \rho(Y, t) = \rho(Y, \langle f \rangle^* X).$$

The other item follows from symmetry. □

2. Contractions

What are (generalized) continuous functions between metric spaces?

Let f be a function, μ and ν be MGNRMs. Provided that they are composable, what does the formula of generalized continuity $f \circ \mu \sqsubseteq \nu \circ f$ mean?

Transforming the formula equivalently, we get:

$$\forall x, z : (f \circ \mu)(x, z) \sqsupseteq (\nu \circ f)(x, z);$$
$$\forall x, z : \mu(\{x\}, \left\langle f^{-1} \right\rangle^{*} \{z\}) \sqsupseteq \nu(fx, z);$$
$$\forall x, z, y \in \left\langle f^{-1} \right\rangle^{*} \{z\} : \mu(x, y) \sqsupseteq \nu(fx, z);$$
$$\forall x, y : \mu(x, y) \sqsupseteq \nu(fx, fy).$$

So generalized continuous functions for metric spaces is what is called *contractions* that is functions that decrease distance.

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