Using The Differential Transformation Method (DTM)

to Solving Ordinary Differential Equations

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Abstract

The differential transformation method (DTM) was first introduced by Zhou 37 years ago. This method is a semi-analytical numerical method for solving homogeneous or inhomogeneous linear ordinary differential equations. Indeed, the differential transform method is based on Taylor series expansion in a different way, in which the differential equation is turned into a recurrence relation to provide a series solution in terms of polynomials.

This research is concerned with the differential transformation method for both ordinary and partial differential equations. To solve initial value and boundary value problems for ordinary differential equations, we use the one-dimensional differential transform technique. Furthermore, we present new modifications to the differential transformation method that improve its algorithm.

The differential transformation method is capable to reduce the size of calculations and handles homogeneous or inhomogeneous linear ordinary differential equations directly. Seven examples are considered for the numerical illustrations of this method. The results demonstrate the reliability and efficiency of this method for such problems.

Keywords: Differential transformation method, ordinary differential equations, Taylor's series expansion.

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1. Introduction

1.1. Overview

Several exact, approximate, and numerical methods, such as the Cauchy-Euler method, exponential sequences, and others, are available for solving differential equations. Most computational methods depend on trial and error or require sophisticated calculations that necessitate numerically approximating results using computers. Recently, a new algorithm called the Differential Transformation Method (DTM) has been proposed for solving linear ODEs, which promises to be faster and more accurate than existing methods. (Ahn, Ramm, & Choi, 2006)

The Differential Transformation Method (DTM) is a numerical technique used for solving differential equations. It is based on the principle of converting differential equations into algebraic equations, thus making it easy to solve the equations. This method was introduced by Zhou in 1986 and has proved to be effective in solving various differential equations, including linear and nonlinear differential and partial differential equations (Ertürk, 2007). Many physical and natural processes in the actual world are expressed as differential equations, most of which are nonlinear. As a result, obtaining accurate or analytical solutions for them is challenging. (Moon, Bhosale, Gajbhiye, & Lonare, 2014)

Many approaches for solving or approximating nonlinear differential equations have been presented. The Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) (Ali, 2007), Homotopy Analysis Method (HAM), and Homotopy Perturbation Method (HPM) are a few examples (Liao, 2004). However, these approaches necessitate computations with some constraints, and in some circumstances, additional terms are required to achieve excellent convergence. There is a need for a technique that can readily handle nonlinear terms with no limits and with fewer computation sizes. Indeed, the so-called Differential Transform Method (DTM), which produces a series solution, can address some of the challenges.

The DTM algorithm has several advantages over other numerical methods for solving linear ODEs. First, it is an analytical method that provides a closed-form solution to the differential equation. This is particularly useful for problems that require high accuracy and precision. Second, the DTM algorithm is highly efficient and can provide solutions quickly and easily. Third, the DTM algorithm can handle a wide range of differential equations, including those with variable coefficients and non-linear terms (Chang & Chang, 2008). The DTM algorithm has been tested on a range of linear ODEs, and the results show that the algorithm is highly accurate and can provide solutions with a high degree of precision. (Diethelm, Ford, J. , & Weilbeer, 2006)

The DTM is a promising new algorithm that offers several advantages over existing numerical methods. Its analytical nature, high accuracy, and ability to handle a wide range of differential equations make it a valuable tool for various applications in science and engineering. With further research and development, the DTM algorithm may become the go-to method for solving linear ODEs. The DTM is a very effective numerical and analytical approach for solving many forms of differential and integral problems. This method turns the differential equations into a recurrence relation, and then, using a different approach, we find convergent series solutions via Taylor series expansion. (İbiş, 2012)

The DTM algorithm is a powerful analytical method that transforms a differential equation into a series of algebraic equations, which can then be solved analytically. The DTM algorithm is based on the idea that a differential equation can be transformed into an infinite series of algebraic equations, which can be efficiently solved using standard algebraic techniques (Chang & Chang, 2008). The DTM algorithm works by transforming the differential equation into a series of Taylor series expansions. The differential transformation transforms the Taylor series expansions into a series of algebraic equations. The algebraic equations can then be solved analytically, producing a closed-form solution to the differential equation. (Gülsu & Sezer, 2006)

In this research, we will make an illustration of how to reach the transformed function from the original function, which includes the fundamental operations, in addition to presenting new examples of differential equations and examining the DTM to find the approximate analytical solution of linear ordinary differential equations, which was performed originally by (Batiha & Batiha, 2011).

1.2. Basic Background

Brief basic background on Differential Equations (DE), Initial Value Problems (IVP), Boundary Value Problems (BVP), Ordinary Differential Equations (ODE), The Taylor Series Method, Convergence Rate, Homogeneous ODE, Kronecker delta as a function, Leibnitz formula and Classification of the Differential Equations is provided in this section.

Definition 1: Differential Equation (DE): “is an equation that contains one or more derivatives of an unknown function” (Trench, 2013). Or we can define it as a mathematical equation that relates a function or a set of functions to their derivatives (i.e., rates of change) with respect to one or more independent variables. In other words, it describes how a quantity changes over time or space, based on the rate at which it changes.

Definition 2: Initial Value Problems (IVP): It is a type of differential equation problem that involves finding a solution to a differential equation that satisfies certain initial conditions. Specifically, an IVP consists of a differential equation, an initial value, and a domain of the independent variable.

The differential equation provides a relationship between the unknown function and its derivatives, while the initial value specifies the value of the function at a given point in the domain. The domain of the independent variable typically represents time or space and specifies the range of values over which the problem is to be solved.

They are often written as:

$$\acute{y} = \mathrm{f}(x, y) \quad \Rightarrow \quad y(a) = b$$

The point (x, y) is where the function $y(x)$ must pass.

The solution to an IVP is a function that satisfies both the differential equation and the initial condition. In general, the solution to an IVP is unique, provided that certain conditions on the differential equation and the initial value are met. Initial value problems arise in many areas of science and engineering, where they are used to model and analyze the behavior of systems that evolve over time or space (Trench, 2013).

Definition 3: Boundary Value Problems (BVP): It is a type of differential equation problem that involves finding a solution to a differential equation subject to certain boundary conditions. Specifically, a BVP consists of a differential equation, a set of boundary conditions, and a domain of the independent variable. The differential equation provides a relationship between the unknown function and its derivatives, while the boundary conditions specify the values of the function at the endpoints of the domain (Hassan, 2008).

Boundary conditions can take many forms, but typically they specify either the value of the function, its derivative, or a combination of both, at one or more points in the domain. The domain of the independent variable typically represents time or space and specifies the range of values over which the problem is to be solved.

The solution to a BVP is a function that satisfies both the differential equation and the boundary conditions. In general, the solution to a BVP is not unique and may depend on the specific choice of boundary conditions. Boundary value problems arise in many areas of science and engineering, where they are used to model and analyse the behaviour of systems that are subject to constraints at their endpoints.

Definition 4: Ordinary Differential Equation (ODE): It is a mathematical equation that relates a function to its derivatives. In other words, given a function of one variable, an ODE describes how the function changes as the variable changes. The equation typically involves the function, one or more of its derivatives, and sometimes the variable itself. ODEs are called "ordinary" because they involve only one independent variable. This is in contrast to partial differential equations (PDEs), which involve multiple independent variables (Trench, 2013).

Definition 5: The Taylor Series Method: The Taylor series method is a numerical method for solving ordinary differential equations (ODEs). It works by approximating the solution of an ODE as a power series expansion around a given point. The method is named after the mathematician Brook Taylor (Trench, 2013).

To apply the Taylor series method, we first write the ODE as a series expansion of the form:

$$y(x + h) = y(x) + h\,y'(x) + (h^2 / 2!)\,y''(x) + (h^3 / 3!)\,y'''(x) + \cdots.$$

where $y(x)$ is the solution of the ODE at the point x, $y'(x)$ is the derivative of y with respect to x evaluated at $x, y''(x)$ is the second derivative of y with respect to x evaluated at x, and so on. The parameter h is the step size, which determines the distance between successive points at which the solution is evaluated (Trench, 2013).

The Taylor series method then approximates the solution by truncating the series after a finite number of terms. The resulting approximation is called the Taylor series polynomial, and it can be used to evaluate the solution at any point within the domain of the ODE.

The Taylor series method is an accurate and versatile numerical method, but it can be computationally intensive, especially for high-order ODEs or for large step sizes. Other numerical methods, such as the Runge-Kutta method or the Euler method, are often used in practice for their simplicity and efficiency (Parumasur, Singh, & Singh, 2009).

Definition 6: Convergence Rate: A numerical method for approximating a solution to an initial value problem is said to be convergent if, assuming no round-off errors, the numerical approximation approaches the exact solution as the step size approaches zero. It should also be noted that (IVP) can have a unique solution, no solution, or infinitely many solutions (Lin, Tang, & Chen, 2014).

Definition 7: Homogeneous ODE: It is a type of ODE in which all the terms involving the dependent variable and its derivatives have the same degree. In other words, if the ODE involves a function $y(x)$ and its derivatives of order n, then it is said to be homogeneous if all the terms in the equation have the same degree of $y(x)$ and its derivatives, namely n (Arfken, Harris, & Weber, 2012).

Formally, an n^{th}-order homogeneous ODE can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = 0$$

where $y^{(k)}(x)$ denotes the k^{th} derivative of y with respect to x, and $a_n(x), a_{n-1}, \ldots, a_1(x), a_0(x)$ are continuous functions of x (Boyce & DiPrima, 2004).

The term "homogeneous" comes from the fact that the ODE has a certain type of symmetry, namely that if $y(x)$ is a solution, then so is any multiple of $y(x)$. In other words, if $y(x)$ satisfies the ODE, then so does any function of the form $cy(x)$ (Boyce & DiPrima, 2004).

Homogeneous ODEs arise in many areas of mathematics and physics, and they can often be solved using techniques such as separation of variables, substitution, or Laplace transforms.

Definition 8: Kronecker Delta: The Kronecker delta is a function of two variables, which are generally typically non-negative integers, it is 1 if the variables are equal and 0 otherwise. (Arfken, Harris, & Weber, 2012) It is named after Leopold Kronecker. Defined for indices i and j as:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Definition 9: Leibnitz formula: The product rule is named after Gottfried Wilhelm Leibniz. If f and g are both n-times differentiable functions, then the product is likewise n-times differentiable, and its n^{th} derivative is provided by:

$$\frac{d^n(g(x)h(x))}{dx^n} = \sum_{r=0}^{n} \binom{n}{r} \frac{d^r g(x)}{dx^r} \frac{d^{n-r} h(x)}{dx^{n-r}}$$

1.3. Classification of the Differential Equations:

We discuss various important methods for categorizing differential equations, including type, order, Degree, and linearity.

1.3.1. Classification by Type: An ordinary differential equation has just ordinary derivatives of one or more unknown functions with regard to a single independent variable (ODE). A partial differential equation is an equation that involves the partial derivatives of one or more unknown functions of two or more independent variables (PDE).

1.3.2. Classification by Order: The differential equation order is dictated by the greatest appearing derivative (either ODE or PDE).

Generally, the equation $f\,(x\,, y,\; y',\, y'', \dots \dots, y^n) = 0.$ is an ordinary differential equation of the n^{th} order.

1.3.3. Classification by Linearity: The categorization of differential equations as linear or nonlinear is critical. An n^{th} order ordinary differential equation $f\,(x\,, y,\; y',\, y'', \dots, y^n) = 0$ If f is a linear function of the variables $x\,, y,\; y',\, y'', \dots \dots, y^n$, it is said to be linear. A similar concept applies to partial differential equations.

As a result, not as the nonlinear equation the generic linear ordinary differential equation of order n is defined as:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x) = g(x)$$

2. Principle of Differential Transformation Method (DTM)

This section will introduce the fundamental concept, and definitions. This concept is then used to establish certain basic theorems.

Remark: In this research, we utilize small letters to represent the original function and capital letters to represent the transformed function. The following theorems and proofs found in (Oke, 2017), (Saeed & Rahman, 2011), (Chen, Lin, & Chen, 1996), (Odibat, 2008), (Abbasov & Bahadir, 2005), (Hassn, 2004), (Arkoglu & Ozkl, 2008), and (Arikoglu & Ozkol, 2005). We will assume that α, β, λ, and c are constants and that, and the theorems can be deduced from equations (1) and (2).

Definition: The Differential Transformation (DT) of a function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \qquad (1)$$

Where the original function is $y(x)$, while the transformed function is $Y(k)$. The differential inverse transform of $Y(k)$ is defined as follows:

$$y(x) = \sum_{k=0}^{\infty} Y\,(k)x^k \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k \qquad (2)$$

By substituting equation (1) in (2) we get:

$$Y(k) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y(x)}{dx^k}\bigg|_{x=0} \qquad (3)$$

This indicates that the differential transform notion is derived from Taylor series expansion. We considered the scenario of $x = 0$ in the preceding definition, although it is true for any fixed real integer $x = x_0$.

Theorem 1. Scalar Multiplication**:** If $f(x) = \alpha\, g\,(x)$, then $F[k] = \alpha\, G[k]$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{1}{k!}\frac{d^k \alpha g(x)}{dx^k}\bigg|_{x=0}$$

$$= \frac{1}{k!}\alpha\frac{d^k g(x)}{dx^k}\bigg|_{x=0}$$

$$= \alpha\, G[k] \qquad \blacksquare$$

Theorem 2. Linear combination:

Under Differential Transformation, a linear combination is closed *i.e.*

$$If \quad f(x) = \alpha\, g\,(x) \pm \beta\, h(x)\,, then \quad F[k] = \alpha\, G[k] \pm \beta\, H[k]$$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{1}{k!}\frac{d^k\left(\left(\alpha g(x)\right) + \left(\beta h(x)\right)\right)}{dx^k}\Bigg|_{x=0}$$

$$= \frac{1}{k!}\left[\frac{d^k(\alpha g(x))}{dx^k} + \frac{d^k(\beta h(x))}{dx^k}\right]\bigg|_{x=0}$$

$$= \frac{1}{k!}\alpha\frac{d^k g(x)}{dx^k}\bigg|_{x=0} + \frac{1}{k!}\beta\frac{d^k h(x)}{dx^k}\bigg|_{x=0}$$

$$= \alpha\, G[k] \pm \beta\, H[k] \qquad \blacksquare$$

Theorem 3. The Differential Transformation of $f'(x)$:

If $f(x) = \frac{dg(x)}{dx}$, then $F(k) = (k+1)G(k+1)$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k)=\frac{1}{k!}\frac{d^k\frac{dg(x)}{dx}}{dx^k}\Bigg|_{x=0}$$

$$F(k)=\frac{1}{k!}\frac{d^{k+1}g(x)}{dx^{k+1}}\Bigg|_{x=0}$$

$$F(k)=\frac{(k+1)!}{k!}\left[\frac{1}{(k+1)!}\frac{d^{k+1}g(x)}{dx^{k+1}}\right]\Bigg|_{x=0}$$

$$F(k)=(k+1)G(k+1) \qquad \blacksquare$$

Theorem 4. The Differential Transformation of $f''(x)$:

If $f(x)=\frac{d^2g(x)}{dx^2}$, then $F(k)=(k+1)(k+2)G(k+1)$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k)=\frac{1}{k!}\frac{d^kf(x)}{dx^k}\Bigg|_{x=0}$$

$$F(k)=\frac{1}{k!}\frac{d^k\frac{d^2g(x)}{dx^2}}{dx^k}\Bigg|_{x=0}$$

$$F(k)=\frac{1}{k!}\frac{d^{k+2}g(x)}{dx^{k+2}}\Bigg|_{x=0}$$

$$F(k)=\frac{(k+2)!}{k!}\left[\frac{1}{(k+2)!}\frac{d^{k+2}g(x)}{dx^{k+2}}\right]\Bigg|_{x=0}$$

$$F(k)=(k+2)(k+1)G(k+2) \qquad \blacksquare$$

Theorem 5. The Differential Transformation of $f^n(x)$:

If $f(x)=\frac{d^ng(x)}{dx^n}$, then $F(k)=(k+1)(k+2)\ldots\ldots(k+n)G(k+n)$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{1}{k!}\frac{d^k \frac{d^n g(x)}{dx^n}}{dx^k}\Bigg|_{x=0}$$

$$F(k) = \frac{1}{k!}\frac{d^{k+n} g(x)}{dx^{k+n}}\bigg|_{x=0}$$

$$F(k) = \frac{(k+n)!}{k!}\left[\frac{1}{(k+n)!}\frac{d^{k+n} g(x)}{dx^{k+n}}\right]\bigg|_{x=0}$$

$$F(k) = (k+1)(k+2)\ldots\ldots.(k+n)G(k+n) \qquad \blacksquare$$

Theorem 6. The Polynomial function:

$If\ \ f(x) = x^n$, $then\ F(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$ is Kronecker delta function

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k x^n}{dx^k}\bigg|_{x=0}$$

But from the differentiation rule, we have

$$\frac{d^k x^n}{dx^k} = n(n-1)(n-2)\ldots\ldots.(n-k+1)x^{(n-k)}$$

$$F(k) = \frac{1}{k!}n(n-1)(n-2)\ldots\ldots.(n-k+1)x^{(n-k)}\bigg|_{x=0}$$

$If\ \ n = k, we\ get$

$$F(k) = \frac{1}{k!}k(k-1)(k-2)\ldots\ldots.1\bigg|_{x=0} = 1$$

$If\ n \neq k, and\ x = 0\ we\ get\ \ F(k) = 0$

$Then$: $$F(k) = \delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \quad \blacksquare$$

Theorem 7. The Constant function: $If\ f(x) = 1\ , then \quad F(k) = \delta(k)$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k 1}{dx^k}\bigg|_{x=0}$$

But we know that $1 = x^0$ so we can denote 0 for n in The Polynomial function and have:

$$F(k) = \frac{1}{k!}\frac{d^k x^n}{dx^k}\bigg|_{x=0}, where\ n = 0$$

$Then$: $$F(k) = \delta(k-0) = \delta(k) \quad \blacksquare$$

Theorem 8. The linear function: $If\ f(x) = x\ , then \quad F(k) = \delta(k-1)$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k x}{dx^k}\bigg|_{x=0}$$

But we know that $x = x^1$ so we can denote 1 for n in The Polynomial function and have:

$$F(k) = \frac{1}{k!}\frac{d^k x^1}{dx^k}\bigg|_{x=0}, where\ n = 1$$

$Then$: $$F(k) = \delta(k-1) \quad \blacksquare$$

Theorem 9. Multiplication of two functions:

$$If\ f(x) = g(x)\,h(x), \quad then \quad F(k) = \sum_{r=0}^{k} G(r)\,H(k-r)$$

Proof: Let $f(x)$ be the original function, then from the Leibnitz formula for the n^{th} derivative of a product we have:

$$\frac{d^k(g(x)h(x))}{dx^k} = \sum_{r=0}^{k} \binom{k}{r} \frac{d^r g(x)}{dx^r} \frac{d^{k-r} h(x)}{dx^{k-r}}$$

Now, by using the previous formula the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!} \frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{1}{k!} \left[\sum_{r=0}^{k} \binom{k}{r} \frac{d^k g(x)}{dx^r} \frac{d^{k-r} h(x)}{dx^{k-r}} \right] \Bigg|_{x=0}$$

$$F(k) = \frac{1}{k!} \left[\sum_{r=0}^{k} \frac{k!}{(k-r)!\ r!} \frac{d^r g(x)}{dx^r} \frac{d^{k-r} h(x)}{dx^{k-r}} \right] \Bigg|_{x=0}$$

$$F(k) = \frac{1}{k!} k! \left[\sum_{r=0}^{k} \frac{1}{r!} \frac{d^r g(x)}{dx^r} \frac{1}{(k-r)!} \frac{d^{k-r} h(x)}{dx^{k-r}} \right] \Bigg|_{x=0}$$

$$F(k) = \sum_{r=0}^{k} G(r) H(k - r) \qquad \blacksquare$$

Theorem 10. The Exponential function: $If\ f(x) = e^{\lambda x}, \quad then \quad F(k) = \frac{\lambda^k}{k!}$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!} \frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{1}{k!} \frac{d^k e^{\lambda x}}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{\lambda^k}{k!} e^{\lambda x}\bigg|_{x=0}$$

$$F(k) = \frac{\lambda^k}{k!} \qquad \blacksquare$$

Theorem 11: $If\ f(x) = (1+x)^c,\ then\ F(k) = \frac{c(c-1).......(c-k+1)}{k!}$

Proof: Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by:

$$F(k) = \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{1}{k!}\frac{d^k (1+x)^c}{dx^k}\bigg|_{x=0}$$

$$F(k) = \frac{c\,(c-1)\,(c-2)\,....(c-k+1)(1+x)^{c-k}}{k!}\bigg|_{x=0}$$

$$F(k) = \frac{c\,(c-1)\,(c-2)\,....(c-k+1)}{k!}$$

$$F(k) = C_k^c$$

3. Numerical Applications

3.1. First order ODEs

Application 1: Consider the first-order differential equation $y' - y = 0$

with the initial condition $y(0) = 1$,

We apply DTM, with initial conditions $Y(0) = 1$

$$Y(k+1) = \frac{1}{(k+1)} Y(k)$$

Put $k = 0$, $Y(1) = 1$

Put $k = 1$, $Y(2) = \frac{1}{2} = \frac{1}{2!}$

Put $k = 2$, $Y(3) = \frac{1}{6} = \frac{1}{3!}$

Put $k = 3$, $Y(4) = \frac{1}{24} = \frac{1}{4!}$

Put $k = 4$, $Y(5) = \frac{1}{120} = \frac{1}{5!}$

Put $k = 5$, $Y(6) = \frac{1}{720} = \frac{1}{6!}$

Put $k = 6$, $Y(7) = \frac{1}{5040} = \frac{1}{7!}$

Put $k = 7$, $Y(8) = \frac{1}{40320} = \frac{1}{8!}$ and so on

Therefore, the closed form of the solution can be easily written as:

$$y(x) = \sum_{k=0}^{k=\infty} Y(k)x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n = e^x$$

Which is the exact solution. (Tanriverdi & Ağırağaç, 2018).

Application 2: Consider the first-order differential equation $y' - y = e^x$

with the initial condition $y(0) = 0$,

We apply DTM, with initial conditions $Y(0) = 0$,

$$Y(k+1) = \frac{1}{(k+1)}\left[\, Y(k) + \frac{1}{k!}\right]$$

Put $k = 0 \,, Y(1) = 1$

Put $k = 1 \,, Y(2) = 1$

Put $k = 2 \,, Y(3) = \frac{1}{2} = \frac{1}{2!}$

Put $k = 3 \,, Y(4) = \frac{1}{6} = \frac{1}{3!}$

Put $k = 4 \,, Y(5) = \frac{1}{24} = \frac{1}{4!}$ and so on

Therefore, the closed form of the solution can be easily written as:

$$y(x) = \sum_{k=0}^{k=n} Y(k)x^k = x + x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \frac{1}{4!}x^5 \dots + \frac{1}{n!}x^{n+1}$$

$$= x\left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n\right] = xe^x$$

Which is the exact solution. (Tanriverdi & Ağırağaç, 2018)

3.2. Second-order ODEs

Application 1: Consider a second-order differential equation $y'' + 3y' + 2y = 24$

with the initial condition $y(0) = 10 \,, y'(0) = 0$

We apply DTM, with initial conditions $Y(0) = 10 \,, Y(1) = 0$

$$Y(k+1) = \frac{1}{(k+1)(k+2)}\left[{}^{-}3(k+1)\, Y(k+1) - 2\, Y(k) + 24(\delta)\right]$$

Put $k = 0 \,, Y(1) = 0$

Put $k = 1 \,, Y(2) = 10$

Put $k = 2 \,, Y(3) = {}^{-}10$

Put $k = 3 \,, Y(4) = \frac{70}{12}$ and so on

Therefore, the closed form of the solution can be easily written as (Patil & Khambayat, 2015):

$$y(x) = \sum_{k=0}^{k=2} Y(k)x^k = 10 + 10x^2 - 10x^3 + \frac{70}{12}x^4 + \cdots$$

Application 2: Consider a second-order differential equation $y'' - 2y' + 5y = 0$

with the initial condition $y(0) = {}^{-}1 , y'(0) = 7$

We apply DTM, with initial conditions $Y(0) = {}^{-}1 , Y(1) = 7$

$$Y(k+2) = \frac{1}{(k+1)(k+2)}[2(k+1)\,Y(k+1) - 5\,Y(k)]$$

Put $k = 0 , Y(2) = \frac{19}{2}$

Put $k = 1 , Y(3) = \frac{1}{2}$

Put $k = 2 , Y(4) = \frac{{}^{-}89}{24}$ and so on

Therefore, the closed form of the solution can be easily written as: (Patil & Khambayat, 2015)

$$y(x) = \sum_{k=0}^{k=\infty} Y(k)x^k = {}^{-}1 + 7x + \frac{19}{2}x^2 + \frac{1}{2}x^3 - \frac{89}{24}x^4 + \cdots$$

3.3. Third-order ODEs

Application 1: Consider a third-order differential equation $y''' + 2y'' - y' - 2y = e^x$

with the initial condition $y(0) = 1 , y'(0) = 2 , y''(0) = 0$

We apply DTM, with initial conditions $y(0) = 1 , y'(0) = 2 , y''(0) = 0$

$$Y(k+3) = \frac{1}{(k+1)(k+2)(k+3)}\left[(k+1)Y(k+1) - 2(k+1)(k+2)Y(k+2) + 2Y(k) + \frac{1}{k!}\right]$$

Put $Y(0) = 1$

Put $k = 0 , Y(1) = 2$

Put $k = 1\,, Y(2) = 0$

Put $k = 2\,, Y(3) = \frac{5}{6}$

Put $k = 3\,, Y(4) = -\frac{5}{24}$

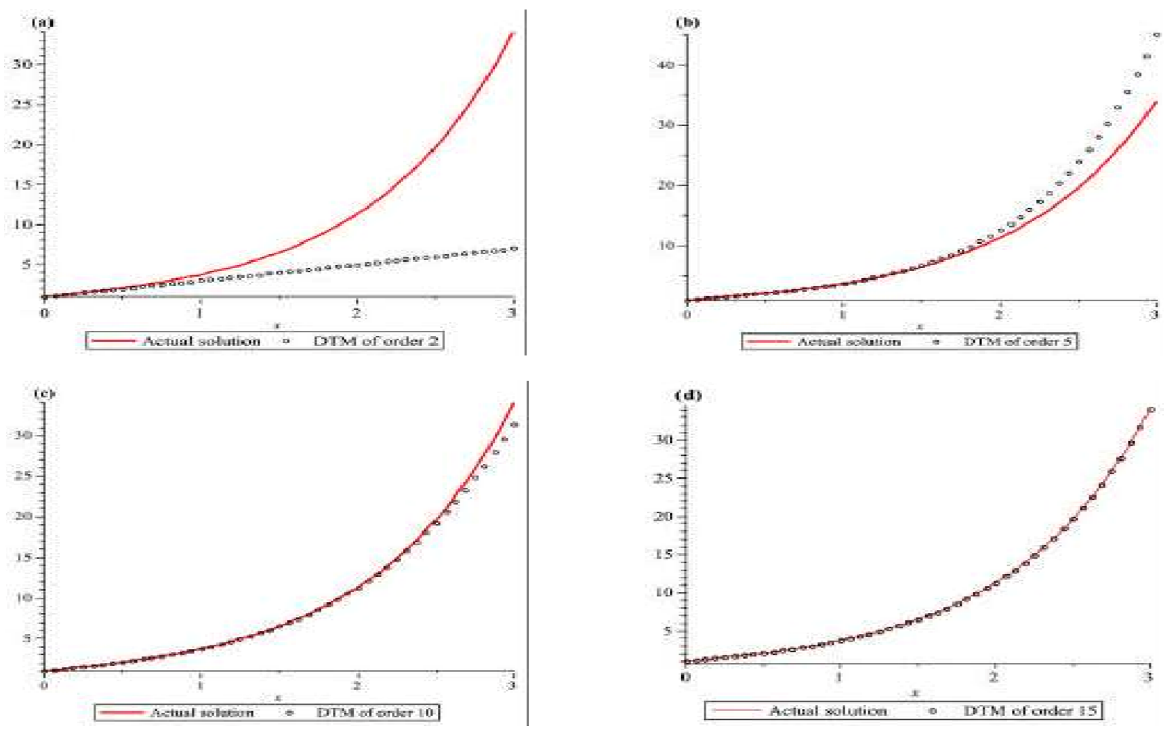
Put $k = 4\,, Y(5) = \frac{12}{15}$

Put $k = 5\,, Y(6) = -\frac{13}{360}$

Put $k = 6\,, Y(7) = \frac{59}{5040}$ and so on

Therefore, the closed form of the solution can be easily written as:

$$y(x) = \sum_{k=0}^{k=\infty} Y(k)x^k = 1 + 2x + \frac{5}{6}x^3 - \frac{5}{24}x^4 + \frac{2}{15}x^5 - \frac{13}{360}x^6 + \frac{59}{5040}x^8 + \cdots$$



This figure compare between DTMs of order 2, 5, 10, and 15 and the actual solution.

The data shown in Figure above clearly demonstrate the high accuracy of DTM of order 15. We will notice that in this differential equation, the higher order of DTM is required to provide an exact solution. (Batiha & Batiha, 2011)

Application 2: Consider a third-order differential equation $y''' + y = 0$ (Agboola, Opanuga, & Gbadeyan, 2015)

with the initial condition $y(0) = 1 , y'(0) = {}^{-}1 , y''(0) = 1$

We apply DTM, with initial conditions $y(0) = 1 , y'(0) = {}^{-}1 , y''(0) = 1$

$$Y(k+3) = -\frac{Y(k)}{(k+3)(k+2)(k+1)}$$

Put $Y(0) = 1$

Put $k = 0 , Y(1) = {}^{-}1$

Put $k = 1 , Y(2) = \frac{1}{2}$

Put $k = 2 , Y(3) = -\frac{1}{6}$

Put $k = 3 , Y(4) = \frac{1}{24}$

Put $k = 4 , Y(5) = -\frac{1}{120}$

Put $k = 5 , Y(6) = \frac{1}{720}$

Put $k = 6 , Y(7) = -\frac{1}{5040}$

Put $k = 7 , Y(8) = \frac{1}{40320}$ and so on

Therefore, the closed form of the solution can be easily written as:

$$y(x) = \sum_{k=0}^{k=\infty} Y(k)x^k = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \frac{x^8}{40320} - \cdots$$

Application 3: Consider a third-order differential equation $y''' - e^x = 0$ (Agboola, Opanuga, & Gbadeyan, 2015)

with the initial condition $y(0) = 3\,, y'(0) = 1\,, y''(0) = 5$

We apply DTM, with initial conditions $y(0) = 3\,, y'(0) = 1\,, y''(0) =$

$$Y(k+3) = \frac{1}{(k+3)!}\frac{1}{k!}$$

Put $Y(0) = 3$

Put $k = 0\,, Y(1) = 1$

Put $k = 1\,, Y(2) = \frac{5}{2}$

Put $k = 2\,, Y(3) = \frac{1}{6}$

Put $k = 3\,, Y(4) = \frac{1}{24}$

Put $k = 4\,, Y(5) = \frac{1}{120}$

Put $k = 5\,, Y(6) = \frac{1}{720}$

Put $k = 6\,, Y(7) = \frac{1}{5040}$

Put $k = 7\,, Y(8) = \frac{1}{40320}$ and so on

Therefore, the closed form of the solution can be easily written as:

$$y(x) = \sum_{k=0}^{k=\infty} Y(k)x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \cdots$$

4. Summary

The Differential Transformation Method (DTM) was effectively used in this research to obtain precise and approximate solutions to the first, second, and third linear differential equations. And some basic theorems of DTM are given and proved.

The approach was used without any linearization, perturbation, or restricted assumptions. As a result, it is not influenced by calculation round-off mistakes and does not necessitate vast amounts of computer memory and time. Unlike other numerical approaches, this method yields a closed-form solution.

This approach has a distinct benefit over any purely numerical method in that it provides a smooth, functional form of the answer across a time step. It is possible to infer that DTM is extremely powerful and efficient in finding analytical and numerical solutions to a large range of linear differential equations.

References

Abbasov, T., & Bahadir, A. R. (2005). The investigation of the transient regimes in the nonlinear systems by the generalized classical method. *Mathematical Problems in Engineering, 2005*(5), 503–519. doi:10.1155/MPE.2005.503

Agboola, O., Opanuga, A., & Gbadeyan, J. (2015). Solution Of Third Order Ordinary Differential Equations Using Differential Transform Method. *Global Journal of Pure and Applied Mathematics, 11*(4), 2511-2516.

Ahn, S., Ramm, A. G., & Choi, U. J. (2006). A scheme for stable numerical differentiation. *Journal of Computational and Applied Mathematics, 186*, 325–334.

Ali, J. (2007). One dimensional differential transform method for some higher order boundary value problems in finite domain. *International Journal of Contemporary Mathematical Sciences, 2*(3), 1493-1504.

Arfken, G. B., Harris, F. E., & Weber, H. J. (2012). *Mathematical Methods for Physicists: A Comprehensive Guide* (7th ed.). Waltham: Academic Press.

Arikoglu, A., & Ozkol, I. (2005). Solution of boundary value problems for integro-differential equations by using differential transform method. *Applied Mathematics and Computation, 168*(2), 1145-1158. doi:10.1016/j.amc.2004.10.009

Arkoglu, A., & Ozkl, I. (2008). Solutions of integral and integro-differential equation systems by using differential transform method. *Computers & Mathematics with Applications, 56*(9), 2411-2417. doi:10.1016/j.camwa.2008.05.017

Batiha, K., & Batiha, B. (2011). A New Algorithm for Solving Linear Ordinary Differential Equations. *World Applied Sciences Journal, 15*(12), 1774-1779.

Boyce, W. E., & DiPrima, R. C. (2004). *Elementary Differential Equations and Boundary Value Problems* (8th ed.). New York: Wiley. Retrieved from mathworld.wolfram.

Chang, S.-H., & Chang, I.-L. (2008). A new algorithm for calculating one-dimensional differential transform of nonlinear functions. *Applied Mathematics and Computation, 195*(2), 799-808. doi:https://doi.org/10.1016/j.amc.2007.05.026

Chen, C.-L., Lin, S.-H., & Chen, C.-K. (1996). Application of Taylor transformation to nonlinear predictive control problem. *Applied Mathematical Modelling, 20*(9), 699-710. doi:10.1016/0307-904X(96)00050-9

Diethelm, K., Ford, J. M., J. , N. F., & Weilbeer, M. (2006). Pitfalls in fast numerical solvers for fractional differential equations. *Journal of Computational and Applied Mathematics*, 482–503.

Ertürk, V. S. (2007). Differential transformation method for solving differential equations of Lane-Emden type. *Mathematical and Computational Applications, 12*(3), 135-139.

Gülsu, M., & Sezer, M. (2006). A Taylor polynomial approach for solving differential-difference equations. *Journal of Computational and Applied Mathematics*(186), 349–364.

Hassan, A.-H. (2008). Application to differential transformation method for solving systems of differential equations. *Applied Mathematical Modelling, 12*(12), 2552-2559. doi:10.1016/j.apm.2007.09.025

Hassn, A.-H. (2004). Differential transformation technique for solving higher-order initial value problems. *Applied Mathematics and Computation, 154*(2), 299-311. doi:10.1016/S0096-3003(03)00708-2

İbiş, B. (2012). Approximate analytical solutions for nonlinear Emden-Fowler type equations by differential. *Department of Main Scieneas, Turkish Air Force Academy*, 1-7.

Liao, S. (2004). *Beyond perturbation : introduction to homotopy analysis method.* Boca Raton: CHAPMAN & HALL/CRC.

Lin, Y., Tang, H.-W., & Chen, C.-K. (2014). Modified Differential Transform Method for Two Singular Boundary Values Problems. *Journal of Applied Mathematics, 2014*(138087), 6. doi:10.1155/2014/138087

Moon, S. D., Bhosale, A. B., Gajbhiye, P. P., & Lonare, G. G. (2014). Solution of Non-Linear Differential Equations by Using Differential Transform Method. *International Journal of Mathematics and Statistics Invention, 2*(3), 78-82.

Odibat, Z. M. (2008). Differential transform method for solving Volterra integral equation with separable kernels. *Mathematical and Computer Modelling, 48*(7), 1144-1149. doi:10.1016/j.mcm.2007.12.022

Oke, A. S. (2017). Convergence of Differential Transform Method for Ordinary Differential Equations. *Journal of Advances in Mathematics and Computer Science, 24*(6), 1-17. doi:10.9734/JAMCS/2017/36489

Parumasur, N., Singh, P., & Singh, V. (2009). Computationally Efficient Technique for Solving ODE Systems Exhibiting Initial and Boundary Layers. *Mathematical Problems in Engineering, 2009*, 15. doi:10.1155/2009/925276

Patil, N., & Khambayat, A. (2015). The Numerical Solution of Differential Transform Method and the Laplace Transform Method for Second Order Differential Equation. *International Journal Of Mathematics And Computer Research, 3*(2), 871-875.

Saeed , R. K., & Rahman, B. M. (2011). Differential Transform Method for Solving System of Delay Differential Equation. *Australian Journal of Basic and Applied Sciences, 5*(4), 201-206.

Tanriverdi, T., & Ağırağaç, N. (2018). Differential Transform Applied To Certain ODE. *Advances in Differential Equations and Control Processes, 19*(3), 213 - 235. doi:10.17654/DE019030213

Trench, W. F. (2013). *Elementary Differential Equations.* San Antonio: Faculty Authored and Edited Books & CDs.

Xie, L.-j., Zhou , C.-l., & Xu, S. (2019). Solving the Systems of Equations of Lane-Emden Type by Differential Transform Method Coupled with Adomian Polynomials. *School of Mathematics and Statistics, 7*(4), 377–392. doi:10.3390/math7040377